A CHARACTERISATION OF VIRTUALLY FREE GROUPS

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Abstract. We prove that a finitely generated group G is virtually free if and only if there exists a generating set for G and $k > 0$ such that all k-locally geodesic words with respect to that generating set are geodesic.

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1. Introduction

A group is called virtually free if it has a free subgroup of finite index.

In this article we characterise finitely generated virtually free groups by the property that a Dehn algorithm reduces any word to geodesic form. Equivalently, a group is virtually free precisely when the set of k-locally geodesic words and the set of geodesic words coincide for suitable k and appropriate generating set.

Let G be a group with finite generating set X. We shall assume throughout this article that all generating sets of groups are closed under the taking of inverses. For a word $w = x_1 \cdots x_n$ over X, we define $l(w)$ to be the length n of w as a string, and $l_G(w)$ to be the length of the shortest word representing the same element as w in G. Then w is called a *geodesic* if $l(w) = l_G(w)$, and a *k-local geodesic* if every subword of w of length at most k is geodesic.

Let $\mathcal{R}$ be a finite set of length-reducing rewrite rules for G; that is, a set of substitutions

$$u_1 \to v_1, u_2 \to v_2, \ldots, u_r \to v_r,$$

where $u_i =_G v_i$ and $l(v_i) < l(u_i)$ for $1 \leq i \leq r$. Then $\mathcal{R}$ is called a *Dehn algorithm* for G over X if repeated application of these rules reduces any representative of the identity to the empty word. It is well-known that a group has a Dehn algorithm if and only if it is word-hyperbolic [1].

More generally (that is, even outside of the group theoretical context), if L is any set of strings over an alphabet X (or, in other words, L is any *language* over X), we shall call L *k-locally excluding* if there exists a finite set F of strings of length at most k such that a string w over X is in L if and only if w contains no substring in F. It is clear that the set of k-local geodesics in a group is k-locally excluding, since we can choose F to be the set of all non-geodesic words of length at most k. We observe in passing that if a set of strings is k-locally excluding then, by definition, it is a k-locally testable and hence locally testable language (see [6]).

We shall say that the group G is k-locally excluding over a finite generating set X when the set of geodesics of G over X is k-locally excluding.

The purpose of this paper is to prove the following theorem.

Theorem 1. *Let G be a finitely generated group. Then the following are equivalent.*

- (i) *G is virtually free.*
- (ii) *There exists a finite generating set X for G and a finite set of length-reducing rewrite rules over X whose application reduces any word over X to a geodesic word; that is G has a Dehn algorithm that reduces all words to geodesics.*
- (iii) *There exists a finite generating set X for G and an integer k such that every k-locally geodesic word over X is a geodesic; that is, G is k-locally excluding over X.*

2. Proof of Theorem 1

The equivalence of (ii) and (iii) is straightforward. Assume (ii), and let $\mathcal{R}$ be a set of length-reducing rewrite rules with the specified property. Let k be the maximal length of a left hand side of a rule in $\mathcal{R}$. Then a k-local geodesic over X cannot have the left hand side of any rule in $\mathcal{R}$ as a subword, and so it must be geodesic. Conversely, assume (iii) and let $\mathcal{R}$ be the set of all rules $u \to v$ in which $l(v) < l(u) \le k$ and $u =_G v$. Then repeated application of rules in $\mathcal{R}$ reduces any word to a k-local geodesic which, by (iii), is a geodesic.

The main part of the proof consists in showing that (i) and (iii) are equivalent. We start with a useful lemma.

Lemma 1. *Let G be a group with finite generating set X, let $k > 0$ be an integer, and suppose that G is k-locally excluding over X. Let w be a geodesic word over X, and let $x \in X$. Then*
(i) $l_G(wx)$ is equal to one of $l(w)+1$, $l(w)$, $l(w)-1$.
(ii) wx is geodesic (that is, $l_G(wx) = l(w)+1$) if and only if vx is geodesic, where v is the suffix of w of length $k-1$ (or the whole of w if $l(w) < k-1$).
(iii) $l_G(wx) - l(w) = l_G(v'x) - l(v')$, where v' is the suffix of w of length $2k-2$ (or the whole of w if $l(w) < 2k-2$).

Proof. The three possibilities for $l_G(wx)$ follow from the fact that w is geodesic and x is a single generator. (ii) is an immediate consequence of G being k-locally excluding. (iii) follows from (ii) when wx is geodesic, so suppose not. Write $w = uv$ with v as defined in (ii), and let z be a geodesic representative of vx. Since v is geodesic, $l(z)$ is either $l(v)$ or $l(v)-1$. In the second case uz is geodesic, so $l_G(wx) - l(w) = l_G(vx) - l(v) = l_G(v'x) - l(v') = -1$

and (iii) follows. In the first case ($l(z) = l(v)$) write $w = u'v''v$ with $v' = v''v$, so $l(v'') = k-1$ provided that u' is non-empty. Now $wx = u'v''vx =_G u'v''z$ where $l(u'v''z) = l(w)$, and either $l_G(wx) = l(u'v''z) = l(w)$ or $l_G(wx) = l(u'v''z) - 1 = l(w) - 1$. So at most one length reduction occurs in the word $u'v''z$, and since $u'v''$ is geodesic, that length reduction must occur, if at all, within the subword $v''z =_G v'x$. Part (iii) follows from this. □

We are now ready to prove that (iii) implies (i) in Theorem 1.

Proposition 1. *Suppose that G is a group with finite generating set X and that the geodesics over X are k-locally excluding for some $k > 0$. Then G is virtually free.*

Proof. We prove this result by demonstrating that the word problem for G can be solved on a pushdown automaton, and then using Muller and Schupp's classification of groups with this property [5].

The automaton to solve the word problem operates as follows. Given an input word w, the automaton reads w from left to right. At any point, the word on the stack is a geodesic representative of the word read so far. Suppose at some point it has u on the stack and then reads a symbol x. It pops $2k-2$ symbols off the stack (or the whole of u if $l(u) < 2k-2$), appends x to the end of the word so obtained, replaces it by a geodesic representative if necessary, and appends that reduced word to the stack. It follows from Lemma 1 that the word now on the stack is a geodesic representative of ux, and hence of the word read so far.

So w represents the identity in G if and only if the stack is empty once all the input has been read and processed, and it follows immediately from [5] that G is virtually free. □

It remains to prove that (i) implies (iii), namely that the set of geodesics of a virtually free group with an appropriate generating set is k-locally excluding for some $k > 0$.

It is proved in [7, Theorem 7.3] that a finitely generated group G is virtually free if and only if it arises as follows: G is the fundamental group of a graph of groups Γ with finite vertex groups $G_1, \dots G_n$, and finite edge groups $G_{i,j}$ for certain pairs $\{i, j\}$.

There are various alternative and equivalent definitions of the fundamental group of a graph of groups, but the one that is most convenient for us is [2, Chapter 1, Definition 3.4]. As is pointed out in [2, Chapter 1, Example 3.5 (vi)], such a group G can be built up as a sequence of groups $1 = H_1, H_2, \dots, H_r = G$, where each H_{i+1} is defined either as a free product with amalgamation (over an edge group) of H_i with one of the vertex groups G_i, or as an HNN extension of H_i with associated subgroups isomorphic to one of the edge groups $G_{i,j}$. The amalgamated free products are done first, building up along a maximal tree, and then the HNN extensions are done for the remaining edges in the graph.

So from now on we shall assume that our virtually free group G can be constructed in this way, where the groups G_i and $G_{i,j}$ are all finite. Hence the result follows from repeated application of the following two lemmas, of which the proofs are very similar.

Notice that the generating set X over which G is k-locally excluding will contain all non-identity elements of each of the vertex groups, G_i and also certain other elements arising from the HNN extensions, which are specified in Lemma 3.

Lemma 2. *Let H be a group which is k-locally excluding over a generating set X for some $k \geq 2$, let K be a finite group, let $A = H \cap K$, and suppose that $A \setminus \{1\} \subset X$.*

Then $G = H *_A K$ is k'-locally excluding over $X' := X \cup (K \setminus A)$, where $k' = 3k - 2$.

Lemma 3. *Let H be a group which is k-locally excluding over a generating set X for some $k \geq 2$, let A and B be isomorphic finite subgroups of H which satisfy $A \setminus \{1\} \subset X$ and $B \setminus \{1\} \subset X$, and let $G = \langle H, t\rangle$ be the HNN extension in which $tat^{-1} = \phi(a)$ for all $a \in A$, where $\phi : A \to B$ is an isomorphism.*

Then G is k'-locally excluding over $X' := X \cup \{ta \mid a \in A\} \cup \{t^{-1}b \mid b \in B\}$, where $k' = 3k - 2$. (Note that the elements of X' in the set $\{t^{-1}b \mid b \in B\}$ are the inverses of those in the set $\{ta \mid a \in A\}$.)

Proof of Lemma 2. Let w be a k'-local geodesic of G over X'. We want to prove that w is geodesic. Suppose not, and let w' be a geodesic word that represents the same element of G. Note that, since $A \setminus \{1\} \subseteq X'$, we cannot have $w \in A$, because that would imply that $l(w) \leq 1$.

We can write $w = w_0k_1w_1k_2 \cdots k_rw_r$, where each $k_i \in K \setminus A$ and each $w_i \in X^*$. Either w_0 or w_r could be the empty word but, since $K \setminus \{1\} \subseteq X'$ and w is a k'-local geodesic with $k' > k \geq 2$, w_i must be non-empty for $0 < i < r$. The 2-locally excluding condition also implies that no non-empty w_i is a word in A^*. In fact, since H is by assumption k-locally excluding over X and $k' > k$, the words w_i are geodesics as elements of H over X, and so the non-empty w_i represent elements of $H \setminus A$.

Similarly, write $w' = w'_0k'_1w'_1k'_2 \cdots k'_{r'}w'_{r'}$.

Now the normal form theorem for free products with amalgamation (see [4, Thm 4.4] or the remark following [3, Chapter 4, Theorem 2.6]) states that, if C is a union of sets of distinct right coset representatives of A in H and in K, then any element of the amalgamated product can be written uniquely as a product of the form $ac_1 \cdots c_s$, where $a \in A$, each $c_i \in C$, and alternate c_i's are in $H \setminus A$ and $K \setminus A$.

Since each $k_i \in K \setminus A$ and each non-empty $w_i \in H \setminus A$, the syllable length s of the group element represented by w is equal to the number of non-trivial words $w_0, k_1, w_1, \ldots, k_r, w_r$, where $c_1 \in H \setminus A$ if and only if w_0 is non-trivial, and $c_s \in H \setminus A$ if and only if w_r is non-trivial. The same applies to w', and hence $r = r'$, w_0 and w'_0 are either both empty or both non-empty, and similarly for w_r and w'_r.

Furthermore, w_r and w'_r are in the same right coset of A in H, and so $w'_r =_H a_r w_r$ for some $a_r \in A$. Then k_r and $k'_r a_r$ are in the same right coset of A in K, and so $k_r =_K b_{r-1} k'_r a_r$ for some $b_{r-1} \in A$. Carrying on in this manner, we can show that there exist $a_i, b_i \in A$ $(0 \le i \le r)$ such that $w'_i =_H a_i w_i b_i$ and $k'_i =_K b_{i-1}^{-1} k_i a_i^{-1}$, where $a_0 = b_r = 1$.

Since $r = r'$ and $l(w') < l(w)$, we must have $l(w'_i) < l(w_i)$ for some i. So one of the words $a_i w_i$, $w_i b_i$, $a_i w_i b_i$ must reduce (in H over X) to a word strictly shorter than w_i.

Suppose first that $w_i b_i$ reduces to a word strictly shorter than w_i. Since $b_r = 1$, we have $i < r$ and so k_{i+1} exists. Then, by Lemma 1, $l_H(v'_i b_i) = l(v'_i) - 1$, where v'_i is the suffix of w_i of length $2k-2$, or the whole of w_i if $l(w_i) < 2k-2$. Now, since $v'_i k_{i+1} =_G (v'_i b_i)(b_i^{-1} k_{i+1})$ with $b_i^{-1} k_{i+1} \in K$, we see that the suffix $v'_i k_{i+1}$ of $w_i k_{i+1}$, which has length at most $2k-1$, is a non-geodesic word in G and, since $2k-1 < k'$, this contradicts the assumption that w is a k'-local geodesic.

The case in which $a_i w_i$ reduces to a word of length less than w_i is similar (here we use a 'mirror image' of Lemma 1), and we find that $i > 0$ and a prefix of $k_i w_i$ of length at most $2k-1$ is non-geodesic, again contradicting the assumption that w is a k'-local geodesic.

It remains to consider the case where the reduction (in H over X) of $a_i w_i b_i$ is strictly shorter than w_i, but each of the reductions of $a_i w_i$ and $w_i b_i$ have the same length as w_i. Since neither a_i nor b_i can be trivial, we have $0 < i < r$, and so k_i and k_{i+1} both exist. We claim that w_i has length at most $3k-4$. For if not, we write $w_i = u' u v'$, where $l(u') = l(v') = k-1$ and $l(u) \ge k-1$, and deduce from Lemma 1 and its mirror image that $a_i w_i b_i =_H yuz$, where $y, z \in X^*$ and $l(y) = l(z) = k-1$. Then since yuz reduces in H over X and H is k-locally excluding over X, some subword of length k must reduce. Such a subword must be a subword of either yu or uz, and so one of $a_i w_i$ or $w_i b_i$ does indeed reduce to a word shorter than w_i, contradicting our assumption. Hence $l(w_i) \le 3k-4$ as claimed.

Now $k_i w_i k_{i+1}$ has length $2 + l(w_i) \le 3k-2$, but $k_i w_i k_{i+1} =_G (k_i a_i^{-1}) w'_i (b_i^{-1} k_{i+1})$ with $k_i a_i^{-1}, b_i^{-1} k_{i+1} \in K$, so $k_i w_i k_{i+1}$ is not a geodesic in G over X', and once again we contradict our assumption that w is a k'-local geodesic. This completes the proof of Lemma 2. □

Proof of Lemma 3. Let w be a k'-local geodesic of G over X'. We want to prove that w is geodesic. Suppose not, and let w' be a geodesic word that represents the same element of G.

Write $w = w_0 t_1^{\epsilon_1} w_1 t_2^{\epsilon_2} w_2 \cdots t_r^{\epsilon_r} w_r$, where each t_i is one of the generators of the form ta ($a \in A$), each ϵ_i is 1 or -1, and each w_i is a word over X. Since $k' > k$, w is a k-local geodesic, so each word w_i is geodesic as an element of H. So if w_i represents a non-trivial element of A or of B, then w_i has length 1. Hence, if $\epsilon_i = 1$ then we cannot have $w_i \in A \setminus \{1\}$, and if $\epsilon_i = -1$ then we cannot have $w_i \in B \setminus \{1\}$, because in those cases $t^{\epsilon_i} w_i$ would be a non-geodesic subword of w of length 2. Also, if w_i is empty with $0 < i < r$, then $\epsilon_i = \epsilon_{i+1}$.

Similarly, write $w' = w_0'(t_1')^{\epsilon_1'} w_1' (t_2')^{\epsilon_2'} w_2' \cdots (t_{r'}')^{\epsilon_{r'}'} w_{r'}'$.

Now the normal form theorem for HNN extensions [3, Chapter 4, Theorem 2.1] states that if C is a union of sets H_A and H_B of distinct right coset representatives of A and of B in H, then any element of the HNN extension G can be written uniquely as a product of the form $ht^{\varepsilon_1} c_1 \cdots t^{\varepsilon_s} c_s$, where $h \in H$, each ε_i is 1 or -1, each $c_i \in C$, and $c_i \in H_A$ or $c_i \in H_B$ when $\varepsilon_i = 1$ or -1, respectively. Also, if $c_i = 1$ with $1 \le i < s$, then $\varepsilon_i = \varepsilon_{i+1}$.

For the normal form of the element of G represented by both w and w', it follows that $r = r' = s$ and $\epsilon_i = \epsilon_i' = \varepsilon_i$ for each i. Furthermore, an inductive argument similar to the one in the proof of Lemma 2 shows that there are elements $a_i, b_i \in A \cup B$ ($0 \le i \le r$) such that $w_i' =_H a_i w_i b_i$ and $(t_i')^{\epsilon_i} = b_{i-1}^{-1} (t_i)^{\epsilon_i} a_i^{-1}$, where $a_0 = b_r = 1$. We have $a_i \in A$ or B when $\epsilon_i = 1$ or -1, respectively, and $b_i \in B$ or A when $\epsilon_{i+1} = 1$ or -1, respectively.

Since $r = r'$ and $l(w') < l(w)$, we must have $l(w_i') < l(w_i)$ for some i. So one of the words $a_i w_i$, $w_i b_i$, $a_i w_i b_i$ must reduce (in H over X) to a word strictly shorter than w_i.

Suppose first that $w_i b_i$ reduces to a word strictly shorter than w_i. Since $b_r = 1$, we have $i < r$ and so t_{i+1} exists. Then, by Lemma 1, $l_H(v_i' b_i) = l(v_i') - 1$, where v_i' is the suffix of w_i of length $2k-2$, or the whole of w_i if $l(w_i) < 2k-2$. Now, since $v_i' t_{i+1}^{\epsilon_{i+1}} =_G (v_i' b_i)(b_i^{-1} t_{i+1}^{\epsilon_{i+1}})$ with $b_i^{-1} t_{i+1}^{\epsilon_{i+1}} \in X'$, we see that the suffix $v_i' t_{i+1}^{\epsilon_{i+1}}$ of $w_i t_{i+1}^{\epsilon_{i+1}}$, which has length at most $2k-1$, is a non-geodesic word in G and, since $2k-1 < k'$, this contradicts the assumption that w is a k'-local geodesic.

The case in which $a_i w_i$ reduces to a word of length less than w_i is similar (using the mirror image of Lemma 1), and we find that $i > 0$ and a prefix of $t_i^{\epsilon_i} w_i$ of length at most $2k-1$ is non-geodesic, again contradicting the assumption that w is a k'-local geodesic.

It remains to consider the case where the reduction (in H over X) of $a_i w_i b_i$ is strictly shorter than w_i, but each of the reductions of $a_i w_i$ and $w_i b_i$ have the same length as w_i. Since neither a_i nor b_i can be trivial, we have $0 < i < r$, and so t_i and t_{i+1} both exist. We claim that w_i has length at most $3k-4$. For if not, we write $w_i = u'uv'$, where $l(u') = l(v') = k-1$ and $l(u) \ge k-1$, and deduce from Lemma 1 and its mirror image that $a_i w_i b_i =_G yuz$, where $y, z \in X^*$ and $l(y) = l(z) = k-1$. Then since yuz reduces in H over X and H is k-locally excluding over X, some subword of length k must reduce.

Such a subword must be a subword of either yu or uz, and so one of $a_i w_i$ or $w_i b_i$ does indeed reduce to a word shorter than w_i, contradicting our assumption. Hence $l(w_i) \leq 3k - 4$ as claimed.

Now $t_i^{\epsilon_i} w_i t_{i+1}^{\epsilon_{i+1}}$ has length $2+l(w_i) \leq 3k-2$, but $t_i^{\epsilon_i} w_i t_{i+1}^{\epsilon_{i+1}} =_G (t_i^{\epsilon_i} a_i^{-1}) w_i' (b_i^{-1} t_{i+1}^{\epsilon_{i+1}})$ with $l_G(t_i^{\epsilon_i} a_i^{-1}) = l_G(b_i^{-1} t_{i+1}^{\epsilon_{i+1}}) = 1$, so $t_i^{\epsilon_i} w_i t_{i+1}^{\epsilon_{i+1}}$ is not a geodesic in G over X', and once again we contradict our assumption that w is a k'-local geodesic. This completes the proof of Lemma 3. □

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