Tame combings, almost convexity and rewriting systems for groups

Susan M. Hermiller*

Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003, USA
smh@mundoe.maths.mu.oz.au

John Meier

Department of Mathematics, Lafayette College, Easton, PA 18042, USA
meierj@lafayette.edu

Abstract: A finite complete rewriting system for a group is a finite presentation which gives an algorithmic solution to the word problem. Finite complete rewriting systems have proven to be useful objects in geometric group theory, yet little is known about the geometry of groups admitting such rewriting systems. We show that a group G with a finite complete rewriting system admits a tame 1-combing; it follows (by work of Mihalik and Tschantz) that if G is an infinite fundamental group of a closed irreducible 3-manifold M, then the universal cover of M is R^3. We also establish that a group admitting a geodesic rewriting system is almost convex in the sense of Cannon, and that almost convex groups are tame 1-combable.

Keywords: Rewriting systems, tame combings, covering conjecture, almost convex groups, isoperimetric inequalities.

* Research supported in part by NSF grants DMS 9022140 at MSRI and INT 9223826 at the University of Melbourne

Tame combings, almost convexity and rewriting systems for groups

1 Introduction

Several properties for finitely presented groups have been defined which can be used to show that a closed P^2-irreducible three-manifold has universal cover homeomorphic to $\mathbf{R}^3$. For example, work of Poénaru [P] shows that if the fundamental group is infinite and satisfies Cannon's almost convexity property, then the universal cover is simply connected at infinity, and hence is $\mathbf{R}^3$ (see [B-T]). Casson later discovered the property C_2, which regrettably is presentation dependent, but which also implies that the universal cover is $\mathbf{R}^3$ [S-G]. Brick and Mihalik generalized the condition C_2 to the quasi-simply-filtered condition [B-M], which is independent of presentation and also implies the covering property. Later Mihalik and Tschantz [M-T] defined the notion of a tame 1-combing for a finitely presented group, which implies the quasi-simply-filtered condition and showed that asynchronously automatic groups and semihyperbolic groups are tame 1-combable.

Using a fairly geometric argument we show that groups with finite complete rewriting systems have tame 1-combings. Using similar techniques, we obtain other geometric properties for groups with finite complete rewriting systems satisfying more restrictive conditions. In particular, a group with a geodesic finite complete rewriting system is almost convex. To finish this circle of ideas we also show that a group which is almost convex for some generating set admits a tame 1-combing. These results establish the following implications:

$$\begin{array}{ccc} \mathcal{GF} & \Longrightarrow & \mathcal{F} \\ \Downarrow & & \Downarrow \\ \mathcal{AC} & \Longrightarrow & \mathcal{T} \end{array}$$

where $\mathcal{GF}$ and $\mathcal{F}$ denote the category of groups admitting geodesic finite complete rewriting systems and groups admitting finite complete rewriting systems, respectively; $\mathcal{AC}$ is the category of almost convex groups; and $\mathcal{T}$ is the category of tame 1-combable groups.

General results about finite complete rewriting systems for fundamental groups of three-manifolds are quite scarce. However, Hermiller and Shapiro [HS] have observed the following theorem concerning three-manifolds which admit a geometric structure. (See [Sc] for more information on these manifolds.)

Theorem. (Hermiller, Shapiro) *Suppose M is a closed three-manifold carrying one of Thurston's eight geometries. In the case that M is hyperbolic, make the further assumption that M is finitely covered by a manifold which fibers over the circle. Then $\pi_1(M)$ has a finite complete rewriting system.*

(According to folklore, Thurston has conjectured that all closed hyperbolic 3-manifolds obey this additional assumption.)

This result follows by combining results in [Sc] with theorems on closure properties for the class of groups admitting finite complete rewriting systems in [G-S1], [G-S2], along with a finite complete rewriting system for surface groups ([L] or [H]).

Any closed irreducible three-manifold satisfying Thurston's geometrization conjecture with infinite fundamental group has universal cover $\mathbf{R}^3$. So in these cases the rewriting systems do not actually give additional information about the covering conjecture. However, since there are computer programs available which search for finite complete rewriting systems for groups, this technique of finding rewriting systems provides a new method for checking the covering property for any three-manifold.

Neither horizontal implication in the diagram above can be reversed. For example, solvgroups admit finite complete rewriting systems but they are not almost convex on any set of generators ([C+]). Therefore solvgroups are examples of groups which have finite complete rewriting systems, but which do not have a geodesic finite complete rewriting system on any generating set. (We note that solvgroups are not contained in any of the categories studied in [M-T]; however, since they do have rewriting systems they are tame 1-combable.)

Many rewriting systems that have been constructed for groups are geodesic because the rules are compatible with a shortlex ordering on words in the generators of the group. Rewriting systems have also been constructed with rules compatible with a weightlex ordering. Gersten [G2] has shown that a group which is almost convex with respect to a finite generating set satisfies a linear isodiametric inequality and an exponential isoperimetric inequality. We show that these inequalities are also satisfied by groups with rewriting systems compatible with a weightlex ordering.

Thus we are able to build another diagram of implications. Below $\mathcal{SF}$ and $\mathcal{WF}$ are the classes of groups admitting shortlex rewriting systems and weightlex rewriting systems, respectively. We use $\mathcal{LE}$ to denote the class of groups with linear isodiametric and exponential isoperimetric inequalities.

$$\begin{array}{ccc} \mathcal{SF} & \Longrightarrow & \mathcal{WF} \\ \Downarrow & & \Downarrow \\ \mathcal{AC} & \Longrightarrow & \mathcal{LE} \end{array}$$

Gersten [G2] has shown that solvgroups satisfy a linear isodiametric inequality, and results in [B] and [E+] show that solvgroups satisfy an exponential isoperimetric inequality. So solvgroups are examples of groups which lie in $\mathcal{LE}$ but not in $\mathcal{AC}$. Regrettably, it is currently unclear which other implications cannot be reversed in this diagram.

The connections between the two diagrams of implications is also unclear. We note that there are groups admitting finite complete rewriting systems which do not lie in $\mathcal{LE}$. Gersten has shown that the group $\langle x, y, z \mid x^y = x^2, y^z = y^2 \rangle$ does not satisfy a linear isodiametric inequality nor an exponential isoperimetric inequality. (See [G1] and [G3] for more details.) Gersten also created the following rewriting system for this group. (A capital letter denotes the inverse of the generator.)

$xX \to 1$	$Xx \to 1$	$yY \to 1$	$Yy \to 1$
$zZ \to 1$	$Zz \to 1$		
$xy \to Xyx$	$xY \to Yxx$	$XXy \to yX$	$XY \to YXX$
$ZY \to YZy$	$zY \to YYz$	$Zyy \to yZ$	$zy \to yyz$

Building on Gersten's work, with help from Brazil, we have established that this rewriting system is complete. Consequently, this group is an example of a group which admits a finite complete rewriting system but does not admit a weightlex rewriting system.

In the next section we give the central definitions and formally state the main theorems. We then proceed to the proofs of each theorem.

2 Definitions and statements of theorems

Given a finite generating set S (which we will assume is closed under inverses, or *symmetric*) for a group G, let $\mathcal{C}_S$ denote the corresponding Cayley graph. There is a natural map from the free monoid on S, S^*, onto G. A *set of normal forms* is a section of this map, that is, it is a choice of how to express each group element in terms of the generators. The corresponding collection of paths in the Cayley graph from the identity out to each vertex described by this set of normal forms is often referred to as a *combing*.

Suppose G is a finitely presented group, and let X be the universal cover of the standard 2-complex associated to some finite presentation. Choose a base point $\epsilon \in X^0$. The following more general definitions of combings are due to Mihalik and Tschantz [M-T].

A 0-combing is essentially a choice of a path in the 1-skeleton X^1 from ϵ to each point of X^0. Viewing X^1 as the Cayley graph of the group G, this can be thought of as the standard notion of a combing for G.

Formally, a *0-combing* of X is a homotopy $\Psi : X^0 \times [0,1] \to X^1$ such that $\Psi(x,0) = \epsilon$ and $\Psi(x,1) = x$ for all $x \in X^0$. The map Ψ is a *tame* 0-combing if for each compact set $C \subseteq X$ there is a compact set $D \subseteq X$ such that for all $x \in X^0$, $\Psi^{-1}(C) \cap (\{x\} \times [0,1])$ is contained in a single path component of $\Psi^{-1}(D) \cap (\{x\} \times [0,1])$.

To define a 1-combing, these definitions are extended to one dimension higher, giving a choice of path in X from ϵ to each point of X^1, in a continuous fashion. A *1-combing* of X is a homotopy $\Psi : X^1 \times [0,1] \to X$ such that $\Psi(x,0) = \epsilon$ and $\Psi(x,1) = x$ for all $x \in X^1$, and such that $\Psi(X^0 \times [0,1]) \subseteq X^1$. It follows that the restriction of Ψ to $X^0 \times [0,1]$ is a 0-combing. The map Ψ is a *tame* 1-combing if this restriction is a tame 0-combing, and if for each compact set $C \subseteq X$ there is a compact set $D \subseteq X$ such that for all edges $e \subseteq X^1$, $\Psi^{-1}(C) \cap (e \times [0,1])$ is contained in a single path component of $\Psi^{-1}(D) \cap (e \times [0,1])$.

A finite complete rewriting system for a finitely presented group G also includes a choice of normal forms for the elements of G. A *rewriting system* consists of a finite set Σ and a subset $R \subseteq \Sigma^* \times \Sigma^*$, where Σ^* is the free monoid on the set Σ. An element $(u,v) \in R$, called a *rule*, is also written $u \to v$. In general if $u \to v$ then for any $x, y \in \Sigma^*$, we write $xuy \to xvy$ and say that the word xuy is *rewritten* (or *reduced*) to the word xvy. for some An element $x \in \Sigma^*$ is *irreducible* if it cannot be rewritten. The ordered pair (Σ, R) is a rewriting system for a monoid M if

$$\langle\ \Sigma \mid u = v \text{ if } (u,v) \in R\ \rangle$$

is a presentation for M. A rewriting system for a group is a rewriting system for the underlying monoid; that is, the elements of Σ must be monoid generators for the group.

A rewriting system (Σ, R) is *complete* if the following conditions hold.

C1) There is no infinite chain $x \to x_1 \to x_2 \to \cdots$ of rewritings. (In this case the rewriting system is called *Noetherian*.)

C2) There is exactly one irreducible word representing each element of the monoid presented by the rewriting system.

The rewriting system is *finite* if R is a finite set. Regrettably, unlike tame combings, the existence of a finite complete rewriting system is dependent upon the choice of generating system.

Because we want to move between the Cayley graph of a group for a given set of generators and rewriting systems based on those generators, all of the rewriting systems in this paper will have generating sets which are closed under taking formal inverses. This often occurs naturally since Σ must be a set of monoid generators, and in most cases this is a harmless restriction since every group which has a finite complete rewriting system on some set of generators Σ, also has one on $\Sigma' = \Sigma \cup \Sigma^{-1}$, with extra rules taking generators in Σ^{-1} to their irreducible representatives in Σ^* from the old rewriting system. For more information on rewriting systems, consult [Co], [H], [L], and the references cited there.

Terminology: In order to improve the exposition we often drop the adjectives "finite" and "complete" when referring to rewriting systems. None the less we are always assuming the rewriting systems are finite complete rewriting systems, and that they have a symmetric set of generators.

Theorem A. *A group which admits a finite complete rewriting system has a tame 1-combing.*

In addition to thinking of the Cayley graph as a topological space, we also think of $\mathcal{C}_S$ as a metric space with the word metric, that is, where each edge is given the metric structure of the unit interval. If n is any positive integer, then we can define the sphere of radius n to be $S(n) = \{\ x \in \mathcal{C}_S \mid d(\epsilon, x) = n\ \}$ where ϵ is the identity element; similarly, the ball of radius n is $B(n) = \{\ x \in \mathcal{C}_S \mid d(\epsilon, x) \leq n\ \}$.

A group G is *almost convex* [Ca] with respect to a finite presentation on a symmetric set of generators if there is a constant A such that for any integer n and any pair of group elements $g, h \in G$ with $g, h \in S(n)$ and $d(g, h) \leq 2$, there is a path in $B(n)$ of length at most A joining g and h.

Similarly we can discuss the geometry of paths; in particular, a path in $\mathcal{C}_S$ is *geodesic* if it is a minimal length path connecting its endpoints. The collection of irreducible words of a finite complete rewriting system is a set of geodesic normal forms if and only if none of the rules of the system increase length; such a rewriting system is a *geodesic* rewriting system. Geodesic rewriting systems, and the special case of shortlex rewriting systems discussed below, are the only instances in this paper where the assumption that the generating set is symmetric might be restrictive. It is conceivable that a group could admit a geodesic (or shortlex) rewriting system, but that no such rewriting system can be created for a symmetric set of generators. No examples of such pathology are known.

In practice one establishes the Noetherian condition C1) for a rewriting system by using a well-founded ordering on the words in Σ^* and checking that if $u \to v \in R$ then $u > v$ in the ordering. For example, one can establish the Noetherian condition by showing that the rewriting rules are compatible with a shortlex ordering. In the shortlex ordering a total order is placed on the generators of G; then a word w is defined to be less than a word v if w is shorter than v or they are of the same length but w is lexicographically prior to v. Rewriting systems that are compatible with a shortlex ordering are geodesic.

Theorem B. *If G has a geodesic finite complete rewriting system, then G is almost convex.*

In order to complete the first circle of implications mentioned in the introduction we establish the following result.

Theorem C. *If G is almost convex, then G has a tame 1-combing.*

An alternative method of establishing the Noetherian property is to use a weightlex ordering. Here each generator is assigned a positive integer weight and w is less than v if the total weight of w is less than the total weight of v, or if the weights are the same, w is lexicographically prior to v. Any group with a finite complete rewriting system compatible with a weightlex ordering, has one on a symmetric set of generators, so again in this case our assumption is not restrictive.

A finitely presented group G satisfies a linear isodiametric inequality if there is a linear function $f : \mathbf{N} \to \mathbf{N}$ such that for any freely reduced word w representing the trivial element of G, there is a van Kampen diagram D with boundary label w and with the (word metric) distance from any vertex of D to the basepoint at most $f(l(w))$, where $l(w)$ is the word length of w. The group G satisfies an exponential isoperimetric inequality if there is an exponential function $g : \mathbf{N} \to \mathbf{N}$ such that for any freely reduced word w representing the trivial element, there is a van Kampen diagram D for w containing at most $g(l(w))$ 2-cells. The reader is referred to [L-S] for information on van Kampen diagrams and to [G4] for background on the isodiametric and isoperimetric inequalities.

Theorem D. *If G admits a finite complete rewriting system compatible with a weightlex ordering, then G satisfies linear isodiametric and exponential isoperimetric inequalities.*

3 Proof of Theorem A

We begin by showing that a rewriting system defines a tame 0-combing for a group.

Lemma 1. *The irreducible words of a finite complete rewriting system for G give a tame 0-combing for G.*

Proof. Given a finite complete rewriting system (Σ, R) for G, let X denote the universal cover of the 2-complex associated to the presentation given by the rewriting system; define a 0-combing using the rewriting system by taking the path $\Psi(x,t) : [0,1] \to X^1$ for a given $x \in X^0$ to be the path following the irreducible word representing x. A prefix u of an irreducible word $w = uv \in \Sigma^*$ cannot contain the left hand side of a rule, so the prefix must also be irreducible. This means that the 0-combing Ψ is *prefix closed*; that is, for any $x \in X^0$, $\Psi(x,t) = \Psi(x,t')$ implies $t = t'$, and for any $y \in \Psi(\{x\} \times (0,1)) \cap X^0$, the combing path $\Psi(\{y\} \times [0,1])$ is a simply a reparametrization following the path $\Psi(\{x\} \times [0,t])$ from ϵ to y.

In [M-T] it is shown that the compact sets in the definitions of tame 0- and 1-combings can be replaced with finite subcomplexes. Given a finite subcomplex $C \subseteq X$, let D be the subcomplex

$$D = C \cup (\cup_{x \in C^0} \Psi(\{x\} \times [0,1])).$$

Then D is also a finite subcomplex. Prefix closure implies that for any point $x \in X^0$, $\Psi^{-1}(D) \cap (\{x\} \times [0,1])$ must be an interval $[0,t]$ for some t. It is then immediate that $\Psi^{-1}(C) \cap (\{x\} \times [0,1])$ is contained in a single path component of the pullback of the set D. Hence Ψ is tame. □

In order to be able to extend the 0-combing from Lemma 1 to a 1-combing, we first replace the rewriting system for G by an equivalent minimal one. For any rewriting system (Σ, R), write $x \xrightarrow{*} y$ whenever $x = y$ or there is a finite chain of arrows $x \to x_1 \to x_2 \to \cdots \to y$. Two complete rewriting systems (Σ, R) and (Σ', R') are *equivalent* if $\Sigma = \Sigma'$, their irreducible words are the same, and whenever $x \xrightarrow{*} z$ in R, with z irreducible, then $x \xrightarrow{*} z$ in R', also. The rewriting system (Σ, R) is *minimal* if for every $(u,v) \in R$, every proper subword of u is irreducible and v is irreducible. The following lemma was proven in [Sq].

Lemma 2. (Squier) *Given a finite complete rewriting system (Σ, R) for G, there is an equivalent minimal finite complete rewriting system.*

Now assume the rewriting system for G is minimal, and let X be the universal cover of the 2-complex associated to this rewriting system. The 1-combing Ψ on $X^1 \times [0,1]$ will be defined inductively.

Given an edge $e \subset X^1$, the endpoints of e correspond to elements $g, h \in G$, where we can write $gs = h$ for some generator $s \in \Sigma$. Write the irreducible word representing a group element g as η_g. Put any total ordering on X^0, for example a shortlex ordering on the irreducible words, and suppose $g > h$ in this ordering. We can consider e to be the union of two oriented edges f_1 and f_2, where the initial vertices are $i(f_1) = g$, $i(f_2) = h$ and the terminal vertices are $t(f_1) = h$, $t(f_2) = g$. A homotopy $\Theta_{f_i} : f_i \times [0,1] \to X$ will be defined for both orientations $i = 1, 2$. These will agree on the endpoints of the edges and will be compatible with the zero combing previously defined. That is, if e is and edge and f_1 and f_2 are the orientations of e, then $\Theta_{f_i}(x,t) = \Psi(x,t)$ whenever x is one of the endpoints $g, h \in X^0$ and $t \in [0,1]$. (We need combings on both orientations in order to mimic the idea of 'prefix closure' used in the proof of Lemma 1.) The 1-combing Ψ restricted to the edge e is defined to be Θ_{f_1} on the corresponding oriented edge f_1.

The *prefix-size* $ps(f)$ of an oriented edge f with $i(f) = g$, $t(f) = h$, and $gs = h$ in G, is defined to be the number of rewritings needed to rewrite $\eta_g s$ to η_h, where at each step the shortest reducible prefix is rewritten. Minimality of R implies that there is only one way to rewrite that prefix in each case, and completeness of R implies that the prefix-size of an oriented edge must always be finite.

We define Θ_f on an oriented edge f by inducting on the prefix-size of f. If $ps(f) = 0$, then $\eta_g s = \eta_h$, so f is in the image of the 0-combing line to h; that is, (abusing notation, and thinking of f as representing the corresponding unoriented edge also) $f \subseteq \Psi(\{h\} \times [0,1])$. For any point $x \in f$, define the path $\Theta_f(\{x\} \times [0,1])$ to be a reparametrization of the path from ϵ to h, stopping at x (see Fig. 1).

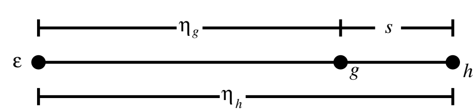

Fig. 1. The prefix size is zero

Now suppose that $ps(f) = 1$. The word $\eta_g s \in \Sigma^*$ contains a suffix which is a left hand side of a rule; this rule is uniquely determined because R is minimal and η_g is irreducible. So we can write $\eta_g = wu$ where $us \to v$ is a rule in R. If the rule $us \to v$ is of the form $s^{-1}s \to 1$, then the unoriented edge corresponding to f is in the image of the 0-combing, and as before we can define Θ_f to be a reparametrization of the 0-combing (see Fig. 2). Otherwise, the relation given by $us = v$ defines the boundary of a 2-cell in X and f is a face of this 2-cell.

The combing paths η_g and η_h from ϵ to the endpoints of f follow a common path along w from ϵ to the 2-cell, and then follow the boundary of the 2-cell. All of the edges in the boundary of this 2-cell except f are in the image of the 0-combing, since they are on the paths given by either η_g or η_h. Θ_f is defined on $f \times [0, 1]$ to follow w and then fill the interior of this cell (see Fig. 3).

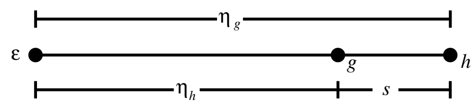

Fig. 2. The prefix size is one, first case

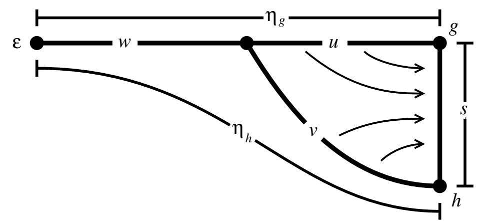

Fig. 3. The prefix size is one, second case

Finally, suppose that $ps(f) = n$, and that for any oriented edge f' with $ps(f') < n$, we have defined $\Theta_{f'}$. As before, the word $\eta_g s \in \Sigma^*$ contains a suffix which is a left hand side of a rule, and this rule is uniquely determined. The map Θ_f is defined on $f \times [\frac{1}{2}, 1]$ to fill the 2-cell defined by the application of this rule. If we write $\eta_g = wu$ in Σ^*, where $us \to v$ is a rule in R, then the image $\Theta_f(x, \frac{1}{2})$ for any point $x \in f$ lies on the paths based at the end of w defined by u or v. Since $wu = \eta_g$ defines the 0-combing path $\Psi(\{g\} \times [0, 1])$, if $\Theta_f(x, \frac{1}{2})$ lies on the path given by u, then $\Theta_f(\{x\} \times [0, \frac{1}{2}])$ can be defined to follow wu from ϵ to $\Theta_f(x, \frac{1}{2})$, by reparametrization (see Fig. 4).

On the other hand, suppose $\Theta_f(x, \frac{1}{2})$ lies on the path defined by v based at the end of w. If $\Theta_f(x, \frac{1}{2}) \in \Psi(X^0 \times [0, 1])$, then again define $\Theta_f(\{x\} \times [0, \frac{1}{2}])$ to follow the 0-combing path from ϵ to $\Theta_f(x, \frac{1}{2})$ with reparametrization. If $\Theta_f(x, \frac{1}{2}) \notin \Psi(X^0 \times [0, 1])$, then $\Theta_f(x, \frac{1}{2})$ must lie on an edge e'. There is an orientation f' of e' with $i(f') = g'$, $t(f') = h'$, and $g's' = h$ for an $s' \in \Sigma$, such that at some stage in the rewriting of $\eta_g s$ by rewriting shortest prefixes, the process must rewrite, in some finite number of steps, a word of the form $\eta_{g'} s' z$ to $\eta_{h'} z$, for some $z \in \Sigma^*$. In particular, this procedure must rewrite the prefix $\eta_{g'} s'$ to $\eta_{h'}$ by always rewriting shortest possible prefixes. This implies that $ps(f') < ps(f)$. By induction, then, the homotopy $\Theta_{f'}$ on $f' \times [0, 1]$ has already been defined; define $\Theta_f(\{x\} \times [0, \frac{1}{2}])$ to follow (with reparametrization) the path defined by the homotopy $\Theta_{f'}$, from ϵ to $\Theta_f(x, \frac{1}{2})$ (see Fig. 5).

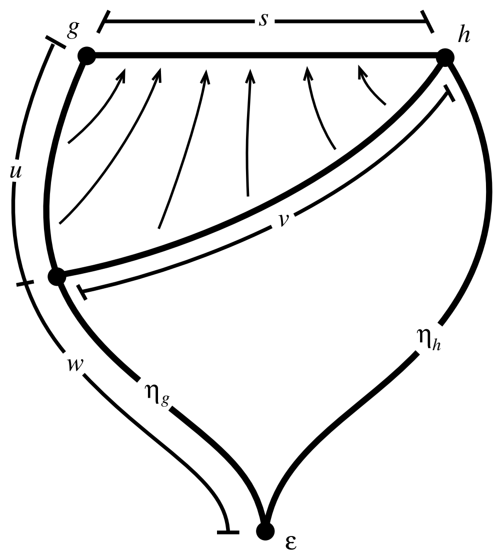

Fig. 4. Defining Θ_f on $[\frac{1}{2}, 2]$

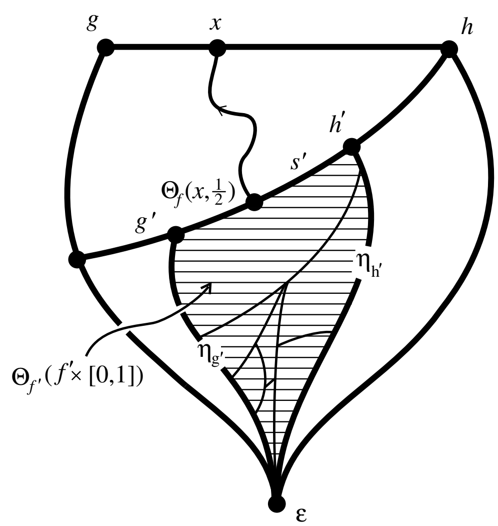

Fig. 5. Completing the definition of Θ_f by induction

We now have homotopies defined for every vertex and for every orientation of every edge in X. The 1-combing Ψ on $X^1 \times [0,1]$ is defined to be the union of 0-combing of Lemma 1 on $X^0 \times [0,1]$ and the edge combings, where the combing for an edge e with endpoints $g > h$ is the homotopy Θ_f on the oriented edge f with $i(f) = g$ and $t(f) = h$. The edge combings are identified along the combings of their bounding vertices.

Proposition 1. *The 1-combing Ψ is a tame 1-combing.*

Proof. Suppose C is a finite subcomplex of X. Let D be the finite subcomplex

$$D = C \cup (\cup_x \Psi(\{x\} \times [0,1])) \cup (\cup_f \Theta_f(f \times [0,1])),$$

where the union is taken over all points $x \in C^0$ and both orientations f of all edges in C^1. To establish that Ψ is tame it is sufficient to show that once the path $\Psi(y,t)$ leaves the subcomplex D it does not return to C; that is, for any $y \in X^1$, $\Psi^{-1}(C) \cap (\{y\} \times [0,1])$ is contained in the component of $\Psi^{-1}(D) \cap (\{y\} \times [0,1])$ containing $(y,0)$.

Assume to the contrary that there is some t such that $\Psi(y,t) \in C$ while there is some $t' < t$ with $\Psi(y,t') \notin D$. By possibly taking a larger t we can assume further that $\Psi(y,t) \in C^1$. If $\Psi(y,t)$ is actually a vertex then, by the construction, $\Psi(\{y\} \times [0,t])$ is the combing path for that vertex, hence $\Psi(\{y\} \times [0,t])$ is contained in D, contradicting the existence of t'.

The only other possibility is that $\Psi(y,t) \in e$ for some edge e in C. But in this case $\Psi(y \times [0,t])$ is contained in $\Theta_{f_1}(f_1 \times [0,1])$ or $\Theta_{f_2}(f_2 \times [0,1])$ where the f_i are the two orientations of e. But both of these sets are contained in D hence once again $\Psi(\{y\} \times [0,t]) \subset D$. □

4 Proof of Theorem B

Suppose (Σ, R) is a geodesic finite complete rewriting system for a group G with symmetric generating set Σ, and suppose that $g, h \in G$ with $g, h \in S(n)$ and $d(g,h) \leq 2$. We will show that there is a path in $B(n)$ of length at most $A = 2\sigma\rho$ joining g and h, where σ is the cardinality of Σ, and ρ is the length of the longest relator in the rewriting system. Write η_g for the irreducible word representing g, as in the proof of Theorem A.

Suppose that $d(g,h) = 2$. Then there is an element $g' \in G$ with $d(g,g') = 1 = d(g',h)$. If $g' \in S(n-1)$, then there is a path of length 2 in $B(n)$ from g to h. If $g' \in S(n)$, we have the same situation as the case when $d(g,h) = 1$, and we will find a path in $B(n)$ from g to g' of length at most $\sigma\rho$; hence there is a path in $B(n)$ from g to h of length at most A. If $g' \in S(n+1)$, let t be the last letter in $\eta_{g'}$; since $\eta_{g'}$ is a geodesic, $g't^{-1} \in S(n)$. A simple modification of the proof for the case when $d(g,h) = 1$ shows that there is a path in $B(n)$ from g to $g't^{-1}$ of length at most $\sigma\rho$ (we leave the details to the reader). Repeating this construction to find a path from $g't^{-1}$ to h gives the desired bound.

Finally, suppose that $d(g,h) = 1$, and let $s_0 \in \Sigma$ be the generator such that $gs_0 = h$. The word $\eta_g s_0$ can be expressed as $w_0 u_0 s_0$ where $u_0 s_0 \to v_0 s_1 \in R$ and

$s_1 \in \Sigma$. Let g_1 be the group element given by the word $w_0 v_0$ so that $h = g_1 s_1$; since rewriting cannot increase length, $g_1 \in B(n)$. If g_1 represents a vertex in the path given by η_h, then we are done. Otherwise we can repeat this process by expressing the word $\eta_{g_1} s_1$ as $w_1 u_1 s_1$ with $u_1 s_1 \to v_1 s_2 \in R$ and letting $g_2 \in B(n)$ be the group element given by the word $w_1 v_1$ so that $h = g_2 s_2$ (see Fig. 6).

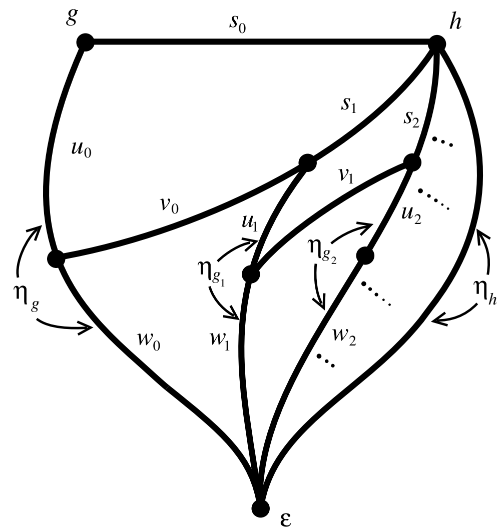

Fig. 6. Rewriting to find the almost convexity constant

If, in this process, it ever occurs that $s_i = s_j$ for $i \neq j$ then the process above would outline how to construct a rewriting from $\eta_{g_i} s_i$ to $\eta_{g_j} s_j$, hence from a word back to itself. By the Noetherian property this is not possible. Thus there can be at most σ steps in this process, and so $\eta_{g_i} s_i = \eta_h$ for some $i < \sigma$. The word $u_0^{-1} v_0 u_1^{-1} v_1 \ldots u_i^{-1} v_i s_i$ then defines a path in $B(n)$ from g to h of length less than $\sigma\rho$. □

5 Proof of Theorem C

Given Poénaru's work [P] and the central motivation for the definition of a tame 1-combing, it surprises us that this theorem has not yet appeared in the literature.

Let G be a finitely presented group with finite presentation $\mathcal{P}$ which is almost convex. Let A be the constant such that between any two vertices g and h in $S(n)$ with $d(g,h) \leq 2$ there is a path in $B(n)$ connecting g to h of length less than or equal to A.

We will work with an extended presentation for G, with the same generators, but with extra relations; we add to the set of relations every word of length $\leq A+2$ which represents the identity. The condition $\mathcal{AC}$ is known to be dependent on the choice of generating set [T], but for a fixed generating set it is independent of choice of relations. We denote the universal cover of the 2-complex associated to this padded presentation by X and define $\tilde{B}(n)$ to be the subcomplex of X consisting of $B(n) \subseteq X^1$ together with all of the 2-cells in X whose boundaries are completely contained in $B(n)$.

Fix a prefix closed geodesic 0-combing; for example, the combing given by taking the shortlex minimal representative of each element. Assume that every edge in the ball $B(n) \subseteq X^1$ has a combing which is consistent with the 0-combing at its endpoints and satisfies the following 'monotone increasing' property:

> (M) For any point $y \in X^1$ for which the 1-combing Ψ is already defined, and any $t \in [0,1]$, if $\Psi(y,t) \in \tilde{B}(k+1) \backslash \tilde{B}(k)$, then $\Psi(y,t') \in X \backslash \tilde{B}(k)$ for all $t' > t$.

We show how to extend this combing to $B(n+1)$.

Let e be an edge in $B(n+1) \backslash B(n)$ corresponding to the generator s and with vertices g and h. Let η_g and η_h be the geodesic combings to the end points of e.

If e is part of the 0-combing then simply reparametrize this combing to give the combing for the edge e. Assume then that e is not part of the 0-combing. There are two possibilities, either (1) both end points of e are in $B(n)$, or (without loss of generality) (2) the last edge t in η_h is also not in $B(n)$. We assume we are in case (2) and leave the easier case (1) to the reader.

Denote the edge from ht^{-1} to $h = gs$ by f. Since the presentation is almost convex, there is a path p in $B(n)$ of length at most A between g and ht^{-1}. Thus the path pts^{-1} is a loop based at the end of the word η_g of length $\leq A+2$, and therefore in X it is filled by a 2-cell. By assumption, the 1-combing has already been defined for all of the edges in the path p, and f is in the image of the 0-combing. The combing for the edge e consists of the union of the combings for the edges in the path pt (identified along the 0-combings for the vertices in pt) plus a mapping from the path pt to e through the 2-cell bounded by pts^{-1}. This extension still satisfies property (M); we define the 1-combing Ψ to be the union of the edge combings identified along the chosen 0-combing.

Proposition 2. *The combing just described is a tame 1-combing.*

Proof. Suppose C is some finite subcomplex of X, and suppose $C \subseteq \tilde{B}(n)$ for some n. Let $D = \tilde{B}(n)$. Because the 1-combing is 'monotone', once a combing line leaves $\tilde{B}(n)$, it never returns, and thus it could not possibly intersect C. In terms of the definition of a tame 1-combing, for all $y \in X^1$, $\Psi^{-1}(C) \cap (\{y\} \times [0,1])$ is contained in the component of $\Psi^{-1}(D) \cap (\{y\} \times [0,1])$ which contains $(y,0)$. □

6 Proof of Theorem D

Let (Σ, R) be a finite complete rewriting system for G such that the rules of the rewriting system are compatible with a weightlex ordering. Define constants $\lambda_h = \max\{\ wt(s_i) \mid s_i \in \Sigma\ \}$ and $\lambda_l = \min\{\ wt(s_i) \mid s_i \in \Sigma\ \}$, and let $\lambda = \lambda_h \lambda_l^{-1}$. Then for any $w \in \Sigma^*$, the length of w is less than or equal to $\lambda_l^{-1} wt(w)$ and the weight of w is at most $\lambda_h l(w)$. So for any $g \in G$, $l(\eta_g) \leq \lambda d(\epsilon, g)$. (Since any subword of an irreducible word is irreducible it follows that the 0-combing defined by the irreducible words is quasi-geodesic.)

Let $w \in \Sigma^*$ be any word evaluating to the identity in G. Then because of the Noetherian property of the rewriting system, w rewrites to the identity. Thus a string of rewritings defines a van Kampen diagram for w with a (possibly degenerate) 2-cell for each application of a rule. Let the rewriting be $w = z_0 \rightarrow z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow 1$ come from rewriting the shortest possible reducible prefix at each stage. Since the rewriting rules are compatible with a weightlex ordering, each rewriting preserves or reduces the weight of the previous word. For a given $k \in \mathbf{Z}$ the total number of words in Σ^* of weight k or less is bounded by the number of words of length at most $k\lambda_l^{-1}$, which in turn is at most $(\sigma + 1)^{k\lambda_l^{-1}}$. Here σ is the cardinality of Σ and the "+1" indicates an empty letter, thus allowing words of length less than $k\lambda_l^{-1}$. Thus the total number of rewriting steps is bounded by $(\sigma + 1)^{wt(w)\lambda_l^{-1}} \leq (\sigma + 1)^{\lambda l(w)}$.

Proposition 3. *If G admits a rewriting system compatible with a weightlex ordering, then G satisfies an exponential isoperimetric inequality.* □

Let g be any vertex in this van Kampen diagram for w; by construction g corresponds to a subword of some word z_i in the rewriting. Because the rewriting of w rewrites the shortest irreducible prefixes, the irreducible word η_g connecting ϵ to g is contained in the diagram for w. Since rewriting can only preserve or reduce weight,

$$wt(\eta_g) \leq wt(z_i) \leq wt(w)$$

Combining this with the bounds given above on lengths, we get

$$l(\eta_g) \leq \lambda_l^{-1} wt(\eta_g) \leq \lambda_l^{-1} wt(w) \leq \lambda l(w).$$

So the distance from ϵ to a vertex in the van Kampen diagram for w is bounded by the linear function $\lambda l(w)$.

Proposition 4. *If G admits a rewriting system compatible with a weightlex ordering, then G satisfies a linear isodiametric inequality.* □

References

[B] Bridson, M.R.: Combings of semidirect products and 3-manifold groups. Geom. Funct. Anal. **3**, 263-278 (1993)

[B-M] Brick, S.G., Mihalik, M.L.: The QSF property for groups and spaces. Math. Z. **220**, 207-217 (1995)

[B-T] Brin, M.G., Thickstun, T.L.: 3-manifolds which are end 1-movable. (Mem. Amer. Math. Soc. vol. 81 no. 411) Providence: American Mathematical Society 1989

[Ca] Cannon, J.W.: Almost convex groups. Geom. Dedicata **22**, 197-210 (1987)

[C+] Cannon, J.W., Floyd, W.J., Grayson, M.A., Thurston, W.P.: Solvgroups are not almost convex. Geom. Dedicata **31**, 291-300 (1989)

[Co] Cohen, D.E.: String rewriting — A survey for group theorists. In: Niblo, G.A., Roller, M.A. (eds.) Geometric group theory, vol. 1. (London Math. Soc. Lecture Note Ser., 181, pp. 37-47) Cambridge: Cambridge University Press 1993

[E+] Epstein, D.B.A., Cannon, J.W., Holt, D.F., Levy, S.V.F., Paterson, M.S., Thurston, W.P.: Word processing in groups. Boston London: Jones and Bartlett 1992

[G1] Gersten, S.M.: The group of the presentation $\mathcal{P} = \langle X, Y, Z \mid X^Y = X^2, Y^Z = Y^2 \rangle$ is not in the class ID. University of Utah (preprint) 1990

[G2] Gersten, S.M.: Isodiametric and isoperimetric inequalities in group extensions. University of Utah (preprint) 1991

[G3] Gersten, S.M.: Dehn functions and l_1-norms of finite presentations. In: Baumslag, G., Miller III, C.F. (eds.) Algorithms and classification in combinatorial group theory. (M.S.R.I. Publications, vol. 23, pp. 195-224) Berlin Heidelberg New York: Springer 1992

[G4] Gersten, S.M.: Isoperimetric and isodiametric functions of finite presentations. In: Niblo, G.A., Roller, M.A. (eds.) Geometric group theory, vol. 1. (London Math. Soc. Lecture Note Ser., 181, pp. 79-96) Cambridge: Cambridge University Press 1993

[G-S1] Groves, J.R.J., Smith, G.C.: Rewriting systems and soluble groups. University of Melbourne (preprint) 1989

[G-S2] Groves J.R.J., Smith, G.C.: Soluble groups with a finite rewriting system. Proc. Edinburgh Math. Soc. **36**, 283-288 (1993)

[H] Hermiller, S.M.: Rewriting systems for Coxeter groups. J. Pure Appl. Algebra **92**, 137-148 (1994)

[L] Le Chenadec, P.: Canonical forms in finitely presented algebras. London: Pitman New York: Wiley 1986

[L-S] Lyndon, R.C., Schupp, P.E.: Combinatorial group theory. (Ergeb. Math., Bd. 89) Berlin Heidelberg New York: Springer 1977

[M-T] Mihalik, M.L., Tschantz, S.T.: Tame combings of groups. Trans. Amer. Math. Soc. (to appear)

[P] Poénaru, V.: Almost convex groups, Lipschitz combing, and π_1^∞ for universal covering spaces of closed 3-manifolds. J. Differential Geom. **35**, 103-130 (1992)

[Sc] Scott, P.: The geometries of 3-manifolds. Bull. London Math. Soc. **15**, 401-487 (1983)

[Sq] Squier, C.C.: Word problems and a homological finiteness condition for monoids. J. Pure Appl. Algebra **49**, 201-217 (1987)

[S-G] Stallings, J.R., Gersten, S.M.: Casson's idea about 3-manifolds whose universal cover is $\mathbf{R}^3$. Internat. J. Algebra Comput. **1**, 395-406 (1991)

[T] Thiel, C.: Zur Fast-Konvexität einiger nilpotenter Gruppen. Bonner Mathematische Schriften 1992