# HNN EXTENSIONS AND STACKABLE GROUPS

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ABSTRACT. Stackability for finitely presented groups consists of a dynamical system that iteratively moves paths into a maximal tree in the Cayley graph. Combining with formal language theoretic restrictions yields auto- or algorithmic stackability, which implies solvability of the word problem. In this paper we give two new characterizations of the stackable property for groups, and use these to show that every HNN extension of a stackable group is stackable. We apply this to exhibit a wide range of Dehn functions that are admitted by stackable and autostackable groups, as well as an example of a stackable group with unsolvable word problem. We use similar methods to show that there exist finitely presented metabelian groups that are non-constructible but admit an autostackable structure.

## 1. INTRODUCTION

Autostackability of finitely generated groups is a topological property of the Cayley graph combined with formal language theoretic restrictions, which is an extension of the notions of automatic groups and groups with finite complete rewriting systems, introduced by Brittenham, Hermiller and Holt in [8]. An autostackable structure for a finitely generated group implies a finite presentation, a solution to the word problem, and a recursive algorithm for building van Kampen diagrams [6]. Moreover, in contrast to automatic groups, Brittenham and Hermiller together with Susse have shown that the class of autostackable groups includes all fundamental groups of 3-manifolds [9], with Holt they have shown autostackable examples of solvable groups that are not virtually nilpotent [7], and with Johnson they show that Stallings' non-$FP_3$ group [8] is autostackable. In analogy with the relationship between automatic and combable groups, removing the formal language theoretic restriction gives the stackable property for finitely generated groups, and stackability implies tame combability [5]. In this paper we give two new characterizations of the stackability property, and determine closure of stackability under HNN extensions. We then apply these results to a variety of examples to exhibit stackable groups that are not algorithmically stackable and to explore the Dehn functions of stackable, algorithmically stackable, and autostackable groups. In the last section we also show that nonconstructible metabelian groups can be autostackable.

To make this more precise, let $G$ be a group with a finite inverse-closed generating set $X$, and let $\Gamma = \Gamma(G, X)$ be the associated Cayley graph.

Denote the set of directed edges in $\Gamma$ by $\vec{E}$, and the set of directed edge paths by $\vec{P}$. For each $g \in G$ and $a \in X$, let $e_{g,a}$ denote the directed edge with initial vertex $g$, terminal vertex $ga$, and label $a$; we view the two directed edges $e_{g,a}$ and $e_{ga,a^{-1}}$ to have a single underlying undirected edge in $\Gamma$.

A *flow function* associated to a maximal tree $\mathcal{T}$ in $\Gamma$ is a function $\Phi : \vec{E} \to \vec{P}$ satisfying the properties that:

(F1) For each edge $e \in \vec{E}$, the path $\Phi(e)$ has the same initial and terminal vertices as $e$.

(F2d) If the undirected edge underlying $e$ lies in the tree $\mathcal{T}$, then $\Phi(e) = e$.

(F2r) The transitive closure $<_\Phi$ of the relation $<$ on $\vec{E}$ defined by
$$e' < e \text{ whenever } e' \text{ lies on the path } \Phi(e) \text{ and the undirected}$$
edges underlying both $e$ and $e'$ do not lie in $\mathcal{T}$,
is a well-founded strict partial ordering.

The flow function is *bounded* if there is a constant $k$ such that for all $e \in \vec{E}$, the path $\Phi(e)$ has length at most $k$. The map $\Phi$ fixes the edges lying in the tree $T$ and describes a "flow" of the non-tree edges toward the tree (or toward the basepoint); starting from a non-tree edge and iterating this function finitely many times results in a path in the tree.

For each element $g \in G$, let $\mathsf{nf}(g)$ denote the label of the unique geodesic (i.e., without backtracking) path in the maximal tree $\mathcal{T}$ from the identity element $\varepsilon$ of $G$ to $g$, and let $\mathcal{N} = \mathcal{N}_\mathcal{T} := \{\mathsf{nf}(g) \mid g \in G\}$ denote the set of these (unique) normal forms. We use functions that pass between paths and words by defining $\mathsf{label} : \vec{P} \to X^*$ to be the function that maps each directed path to the word labeling that path and defining $\mathsf{path} : \mathcal{N} \times X^* \to \vec{P}$ to be $\mathsf{path}(\mathsf{nf}(g), w) :=$ the path in $\Gamma$ that starts at $g$ and is labeled by $w$. Observe that $\mathsf{path}(\mathcal{N} \times X) = \vec{E}$.

**Definition 1.1.** [6, 7] *Let $G$ be a group with a finite inverse-closed generating set $X$.*

(i) *The group $G$ is* stackable *over $X$ if there is a bounded flow function on a maximal tree in the associated Cayley graph. The* stacking map *is*
$$\phi := \mathsf{label} \circ \Phi \circ \mathsf{path} : \mathcal{N}_\mathcal{T} \times X \to X^*.$$

(ii) *The group $G$ is* algorithmically stackable *over $X$ if $G$ admits a bounded flow function $\Phi$ for which the graph*
$$\mathsf{graph}(\phi) := \{(\mathsf{nf}(g), a, \phi(\mathsf{nf}(g), a)) \mid g \in G, a \in X\}$$
*of the stacking map $\phi$ is decidable.*

(iii) *The group $G$ is* autostackable *over $X$ if $G$ has a bounded flow function $\Phi$ for which the graph of the associated stacking map is synchronously regular.*

A stackable group $G$ over a finite generating set $A$ is finitely presented, with finite presentation $R_\Phi = \langle X \mid \{\phi(y, a) = a \mid y \in \mathcal{N}_\mathcal{T}, a \in X\} \rangle$ (called the *stacking presentation*) associated to the flow function $\Phi$.

Each of these three stackability properties can also be stated in terms of prefix-rewriting systems. A stackable structure is equivalent to a bounded complete prefix-rewriting system for $G$ over $X$, for which the irreducible words are exactly the elements of the set $\mathcal{N}_\mathcal{T}$. A group is algorithmically stackable (respectively, autostackable) if and only if it admits a decidable (respectively, synchronously regular) bounded complete prefix-rewriting system [7]. (See Section 2 for definitions of these terms.)

In Section 2, we begin with notation and definitions we will use throughout the paper.

Section 3 contains several characterizations of stackability using properties of their van Kampen diagrams, which we apply in our proofs in later sections of this paper.

Section 4 contains the proof of the following closure property for the class of stackable groups with respect to HNN extensions.

**Theorem 4.4.** *Let $H$ be a stackable group, let $A, B \leq H$ be finitely generated, and let $\psi : A \to B$ be an isomorphism. Then the HNN extension $G = H*_\psi$ is also stackable.*

Corollary 4.6 addresses closure of auto- and algorithmic stackability of HNN extensions with additional constraints. In the case of algorithmically stackable groups, HNN extension closure can also be stated in terms of a decision problem. For a group $H$ with a finite inverse-closed generating set $Y$ and subgroup $A$, the *subgroup membership problem* is decidable if there is an algorithm that, upon input of any word $w$ over $Y$, determines whether or not $w$ represents an element of $A$.

**Corollary 4.7.** *Let $H$ be an algorithmically stackable group, let $A, B \leq H$ be finitely generated, and let $\psi : A \to B$ be an isomorphism. Suppose further that the subgroup membership problem is decidable for the subgroups $A$ and $B$ in $H$. Then the HNN extension $G = H*_\psi$ is also algorithmically stackable.*

We give three applications of Theorem 4.4 in Section 5. In the first, we apply Mihailova's [17] construction of subgroups of direct products of free groups with unsolvable subgroup membership problem, to show that stackability and autostackability are not the same property.

**Theorem 5.1.** *There exists a stackable group with unsolvable word problem, and hence stackability does not imply algorithmic stackability.*

Consequently the class of stackable groups includes groups whose Dehn function is not computable. Our second application is a proof that the hydra groups of Dison and Riley [12] are algorithmically stackable, and consequently algorithmically stackable groups admit extremely large Dehn functions.

**Theorem 5.2.** *The class of algorithmically stackable groups includes groups with Dehn functions in each level of the Grzegorczyk hierarchy of primitive recursive functions.*

For the third application in Section 5, we we study the Baumslag-Gersten group [2] (also known as the Baumslag group), which is a (nonmetabelian) HNN extension of a Baumslag-Solitar group (which is autostackable [7]).

**Theorem 5.3.** *The Baumslag-Gersten group $\langle a, s \mid (sas^{-1})a(sa^{-1}s^{-1}) = a^2 \rangle$ is autostackable.*

Platonov [18] (and in the case of the lower bound, Gersten [13]) has shown that the Dehn function of the Baumslag-Gersten group is the nonelementary function $n \rightarrow \mathsf{tower}_2(\log_2(n))$, where $\mathsf{tower}_2(1) = 2$ and $\mathsf{tower}_2(k) = 2^{\mathsf{tower}_2(k-1)}$. Although it is an open question whether the Baumslag-Gersten group, or any other group with nonelementary Dehn function, can have a finite complete rewriting system, these results show that groups with a bounded synchronously regular complete prefix-rewriting system admit such Dehn functions.

**Corollary 5.4.** *The class of autostackable groups includes groups with nonelementary Dehn functions.*

Finally, in Section 6, we consider metabelian groups. Groves and Smith [14] showed that a metabelian group $G$ has a finite complete rewriting system if and only if $G$ has the homological finiteness condition $FP_\infty$, and also if and only if $G$ is constructible (that is, $G$ can be obtained from finite groups by iteratively taking finite extensions, amalgamations, and finite rank HNN extensions). Since synchronously regular bounded prefix-rewriting systems (i.e., autostackable structures) are an extension of finite complete rewriting systems, it is natural to ask whether autostackability is also equivalent to $FP_\infty$ and constructibility in metabelian groups.

We consider Baumslag's [3] example of a finitely presented metabelian group whose commutator subgroup has infinite rank. This group is finitely presented, and hence has homological type $FP_2$, but the rank of the commutator subgroup implies that the group is not constructible, and hence is not of type $FP_\infty$. For $p < \infty$, the $p$-torsion analog of Baumslag's metabelian group is the Diestel-Leader group $\Gamma_3(p)$, whose Cayley graph with respect to a certain finite generating set is the Diestel-Leader graph $DL_3(p)$; for details and more information, see the paper of Stein, Taback, and Wong [19] and the references there. The Diestel-Leader groups are finitely presented metabelian groups that are not of type $FP_3$ (see [1]), and hence also are nonconstructible metabelian groups. Each of these groups can be realized as an HNN extension, but since the base group of this extension is not finitely generated, Theorem 4.4 of Section 4 does not apply in this case, and new methods are developed in Section 6 to show the following.

**Theorem 6.1.** *Baumslag's metabelian group $G_\infty = \langle a, s, t \mid a^s = a^t a, [a^t, a] = 1, [s, t] = 1 \rangle$ is algorithmically stackable, and the Diestel-Leader torsion analogs $G_p = \langle a, s, t \mid a^s = a^t a, [a^t, a] = 1, [s, t] = 1, a^p = 1 \rangle$ with $p \in \mathbb{Z}$ and $p \geq 2$ are autostackable.*

This shows that there exist nonconstructible metabelian groups that admit a synchronously regular bounded prefix-rewriting systems, giving a negative answer to the question above.

**Corollary 6.2.** *The class of autostackable groups contains nonconstructible metabelian groups.*

## 2. Notation and background

Throughout this paper, let $G$ be a group with a finite *symmetric* generating set $X$; that is, such that the generating set $X$ is closed under inversion. Throughout the paper we assume that no element of $X$ represents the identity element of $G$.

Let $X^*$ denote the set of all words over $X$, and let $X^+$ denote the set of all words except the empty word 1. A set $\mathcal{N}$ of *normal forms* for $G$ over $X$ is a subset of $X^*$ such that the restriction of the canonical surjection $\rho : X^* \to G$ to $\mathcal{N}$ is a bijection. As in Section 1, the symbol $\mathsf{nf}(g)$ denotes the normal form for $g \in G$. By slight abuse of notation, we use the symbol $\mathsf{nf}(w)$ to denote the normal form for $\rho(w)$ whenever $w \in X^*$.

For a word $w \in X^*$, we write $w^{-1}$ for the formal inverse of $w$ in $X^*$, and let $l(w)$ denote the length of the word $w$. For words $v, w \in X^*$, we write $v = w$ if $v$ and $w$ are the same word in $X^*$, and write $v =_G w$ if $v$ and $w$ represent the same element of $G$.

Let $\varepsilon$ denote the identity of $G$. For $g, h \in G$, we use $g^h$ to denote the conjugate $hgh^{-1}$ of $g$.

A *symmetrized* presentation $\mathcal{P} = \langle X \mid R \rangle$ for $G$ satisfies the properties that the generating set $X$ is symmetric and the set $R$ of defining relations is closed under inversion and cyclic conjugation. Let $C$ be the Cayley 2-complex corresponding to this presentation, whose 1-skeleton $C^1 = \Gamma$ is the Cayley graph of $G$ over $A$. For $g \in G$ and $x \in X$, let $e_{g,x}$ denote the edge of $\Gamma$ labeled by $x$ with initial vertex $g$. We consider the two directed edges $e_{g,x}$ and $e_{gx,x^{-1}}$ to have the same underlying directed edge in $\Gamma$ between the vertices $g$ and $gx$.

2.1. **Diagrams.** For an arbitrary word $w$ in $X^*$ that represents the trivial element $\varepsilon$ of $G$, there is a *van Kampen diagram* $\Delta$ for $w$ with respect to $\mathcal{P}$. That is, $\Delta$ is a finite, planar, contractible combinatorial 2-complex with edges directed and labeled by elements of $X$, satisfying the properties that the boundary of $\Delta$ is an edge path labeled by the word $w$ starting at a basepoint vertex $*$ and reading counterclockwise, and every 2-cell in $\Delta$ has boundary labeled by an element of $R$. (Note that we do not assume that van Kampen diagrams in this paper are reduced; that is, we allow adjacent 2-cells in $\Delta$ to be labeled by the same relator with opposite orientations.)

For any van Kampen diagram $\Delta$ with basepoint $*$, let $\pi_\Delta : \Delta \to C$ denote a cellular map such that $\pi_\Delta(*) = \varepsilon$ and $\pi_\Delta$ maps edges to edges preserving both label and direction. Given $w \in X^*$, we denote by $w\Delta$ the diagram obtained by gluing the terminus of a path labeled by $w$ to the basepoint $*$ of $\Delta$.

2.2. **Rewriting systems and languages.** The *regular* languages over a finite set $X$ are the subsets of $X^*$ obtained from the finite subsets of $X^*$ using finitely many operations from among union, intersection, complement, concatenation ($S \cdot T := \{vw \mid v \in S \text{ and } w \in T\}$), and Kleene star ($S^0 := \{1\}$, $S^n := S^{n-1} \cdot S$ and $S^* := \cup_{n=0}^\infty S^n$). Equivalently, a subset $L \subseteq X^*$ is regular if there is a monoid homomorphism $\gamma : X^* \to M$ for some finite monoid $M$, such that $L$ is the preimage $L = \gamma^{-1}(S)$ for a subset $S$ of $M$. Also equivalently, a language $L \subseteq X^*$ is regular if $L$ is the language accepted by a finite state automaton.

A subset $L \subseteq (X^*)^n$ is called a *synchronously regular* language if the padded extension set $\{\mathsf{pad}(w) \mid w \in L\}$ is a regular language over the finite alphabet $(X \cup \{\$\})^n$ (with $\$ \notin X$) where $\mathsf{pad}(a_{1,1} \cdots a_{1,m_1}, ..., a_{n,1} \cdots a_{n,m_n}) := ((a_{1,1}, ..., a_{n,1}), ..., (a_{1,N}, ..., a_{n,N}))$ for $N = \max\{m_i\}$ whenever $a_{i,j} \in X$ for all $1 \le i \le n$ and $1 \le j \le m_i$ and $a_{i,j} = \$$ otherwise.

The class of regular languages is closed under both image and preimage via monoid homomorphisms and under quotients, and the class of synchronously regular languages is closed with respect to finite unions and intersections, Cartesian products, and projection onto a single coordinate.

A language $L \subseteq X^*$ is *decidable*, also known as *recursive*, if there is a Turing machine that, upon input of any word $w$ over $X$, determines (in a finite amount of time) whether or not $w \in L$. The class of decidable languages is also closed under union, intersection, complement, concatenation, Kleene star, and image via monoid homomorphisms (that map nonempty words to nonempty words).

See [10] and [15] for more information about regular, synchronously regular, and decidable languages.

A *complete prefix-rewriting system* for a group $G$ consists of a set $X$ and a set of rules $R \subseteq X^* \times X^*$ (with each $(u, v) \in R$ written $u \to v$) such that $G$ is presented (as a monoid) by $G = Mon\langle X \mid u = v \text{ whenever } u \to v \in R \rangle$, and the rewritings $uy \to vy$ for all $y \in X^*$ and $u \to v$ in $R$ satisfy:

(1) there is no infinite chain $w \to x_1 \to x_2 \to \cdots$ of rewritings, and
(2) each $g \in G$ is represented by exactly one irreducible word over $X$.

The prefix-rewriting system is *bounded* if $X$ is finite and there is a constant $k$ such that for each pair $(u, v)$ in $R$ there are words $s, t, w \in X^*$ such that $u = ws$, $v = wt$, and $l(s) + l(t) \le k$. The prefix-rewriting system is *synchronously regular* if the set $X$ is finite and the set of rules $R$ is synchronously regular.

A *finite complete rewriting system* for a group $G$ is a finite set $R' \subseteq X^* \times X^*$ presenting $G$ as a monoid, such that the rewritings $xuy \to xvy$ for all $x, y \in X^*$ and $u \to v$ in $R'$ satisfy (1) and (2) above. Any finite complete rewriting system $R'$ has an associated synchronously regular bounded complete prefix-rewriting system given by $R = \{xu \to xv \mid x \in X^*, (u, v) \in R'\}$.

## 3. The fully $\mathcal{N}$-triangular and $\mathcal{N}$-labeled properties

In this section we develop several conditions that are equivalent to stackability, which will be used to simplify proofs in later sections of this paper.

Let $G$ be a group with a finite symmetrized presentation $\mathcal{P} = \langle X \mid R \rangle$. Let $\mathcal{N}$ be a prefix-closed set of normal forms for $G$ over $X$. For each element $g$ in $G$, let $\mathsf{nf}(g)$ denote the element of $\mathcal{N}$ representing $g$. Let $\mathcal{T}_{\mathcal{N}}$ denote the tree in the Cayley graph $\Gamma$ for $G$ over $A$ consisting of the underlying undirected edges that lie along paths from the identity vertex $\varepsilon$ labeled by elements of $\mathcal{N}$.

**Definition 3.1.** We say that a van Kampen diagram $\Delta$ is $\mathcal{N}$-*triangular* if $\partial\Delta$ is labeled by a word of the form $\mathsf{nf}(g)x\mathsf{nf}(gx)^{-1}$ with $g \in G$ and $x \in X$. We refer to the paths $p_{lower}$, $p_x$, $p_{upper}$ in $\partial\Delta$ labeled by $\mathsf{nf}(g)$, $x$, and $\mathsf{nf}(gx)$ as the *lower normal form*, *isolated edge*, and *upper normal form* of $\Delta$, respectively.

**Definition 3.2.** A $\mathcal{N}$-triangular van Kampen diagram $\Delta$ is $\mathcal{N}$-*labeled* if for every vertex $v$ in the 0-skeleton $\Delta^{(0)}$ of $\Delta$, there is a path in $\Delta$ from the basepoint to $v$ labeled by the normal form $\mathsf{nf}(\pi_\Delta(v))$. In this case $\Delta$ determines a set of normal forms starting at the basepoint, namely

$$\mathsf{nf}(\Delta) := \{\mathsf{nf}(\pi_\Delta(v)) \mid v \in \Delta^{(0)}\}.$$

If $G$ is stackable over the normal form set $\mathcal{N}$, with stacking relations contained in $R$, then there is a recursive procedure for building van Kampen diagrams for $G$ over the presentation $\mathcal{P}$; see [6] for details of this *stacking procedure*. In the following recursive definition we describe a similar property.

**Definition 3.3.** Fully $\mathcal{N}$-triangular diagrams are recursively defined as follows. A diagram $\Delta$ is called

  i) *degenerate* if $\Delta$ has no 2-dimensional cells.
  ii) *minimal* if $\Delta$ has a single 2-dimensional cell $\sigma$ and $p_x \subset \partial\sigma$.
  iii) *fully $\mathcal{N}$-triangular* if $\Delta$ is either degenerate, minimal, or there is a 2-dimensional cell $\sigma$ in $\Delta$ with $p_x \subset \partial\sigma$, which we call the *isolated cell*, satisfying the following property. If $e_1, ..., e_t$ are the successive edges of the path in $\partial\sigma \smallsetminus p_x$ from the initial vertex to the terminus of the edge $p_x$, then for each $i = 1, ..., t$ there is a fully $\mathcal{N}$-triangular van Kampen diagram $\Delta_i \subseteq \Delta$ having $e_i$ as isolated edge and the same basepoint as $\Delta$, such that
    - for each $i$, $\Delta_i \cap \Delta_{i+1}$ is both the upper normal form of $\Delta_i$ and the lower normal form of $\Delta_{i+1}$, and
    - $\Delta$ is the disjoint union of the $\Delta_i$ and $\sigma$, with the $\Delta_i$ glued along these successive normal forms, and $\sigma$ glued to the $\Delta_i$ along the edges $e_i$.

In this case, $\Delta$ determines a set of fully $\mathcal{N}$-triangular van Kampen diagrams, namely, the $\Delta_i$ together with the set of fully $\mathcal{N}$-triangular van Kampen diagrams determined by them; we denote this set by $\mathrm{ft}(\Delta)$.

We note that every fully $\mathcal{N}$-triangular van Kampen diagram is also $\mathcal{N}$-labeled. Also note that if $w \in X^*$ and $\Delta$ is $\mathcal{N}$-labeled or fully $\mathcal{N}$-triangular,

and moreover if for every $\mu \in \mathsf{nf}(\Delta)$ we have $w\mu \in \mathcal{N}$, then $w\Delta$ is also $\mathcal{N}$-labeled or fully $\mathcal{N}$-triangular, respectively.

In the case that the group $G$ is stackable with respect to $\mathcal{N}$, let $\Delta_{g,x}$ denote the van Kampen diagram with boundary $\mathsf{nf}(g)x\mathsf{nf}(gx)^{-1}$ obtained using the stacking procedure. Given a directed edge $e_{g,x}$ of the Cayley graph $\Gamma$, if $e_{g,x}$ lies in the tree $\mathcal{T}_{\mathcal{N}}$, then $\Delta_{g,x}$ is degenerate, and so is fully $\mathcal{N}$-triangular. To check that for any edge $e_{g,x}$ not in $\mathcal{T}_{\mathcal{N}}$ the diagram $\Delta_{g,x}$ is fully $\mathcal{N}$-triangular, we note that the isolated cell has boundary $\Phi(e_{g,x})$ and $p_x$, and the recursive property in Definition 3.3 iii) holds true by induction using the partial order $<_\Phi$ on the set of recursive edges given by the stacking system. In fact, if $e_{h,y} <_\Phi e_{g,x}$, then $e_{h,y}$ is an edge needed in the process to transform $\mathsf{nf}(g)x$ into its normal form in the stacking reduction procedure, and $\Delta_{h,y} \in \mathrm{ft}(\Delta_{g,x})$.

**Definition 3.4.** A *stackable system of fully $\mathcal{N}$-triangular van Kampen diagrams* (respect to $\mathcal{P}$) is a set

$$\mathcal{S} := \{\Delta_{g,x} \mid g \in G, x \in X\}$$

of fully $\mathcal{N}$-triangular diagrams, one for each $g \in G$ and $x \in X$, such that the boundary of the diagram $\Delta_{g,x}$ is labeled by $\mathsf{nf}(g)x\mathsf{nf}(gx)^{-1}$, the diagram $\Delta_{g,x}$ is degenerate if and only if $e_{g,x}$ is in the tree $\mathcal{T}_{\mathcal{N}}$, and whenever $\Delta_{g,x} \in \mathcal{S}$ contains more than one 2-cell, the associated subdiagrams $\Delta_i$ in Definition 3.3 iii) also belong to $\mathcal{S}$.

**Proposition 3.5.** *The following are equivalent for a finitely presented group $G = \langle X \mid R \rangle$ with a prefix-closed normal form set $\mathcal{N}$ over $X$:*

- i) *$G$ is stackable with respect to $\mathcal{N}$.*
- ii) *There is a stackable system of fully $\mathcal{N}$-triangular van Kampen diagrams.*
- iii) *For every $g \in G$ and $x \in X$ there is a fully $\mathcal{N}$-triangular van Kampen diagram $\Delta_{g,x}$ with boundary $\mathsf{nf}(g)x\mathsf{nf}(gx)^{-1}$.*
- iv) *For every $w \in \mathcal{N}$ and $x \in X$ there is a $\mathcal{N}$-labeled van Kampen diagram $\Delta'_{w,x}$ with boundary $wx\mathsf{nf}(wx)^{-1}$.*

*Proof.* The fact that i) implies ii) follows from the discussion above, and the implications ii) $\Rightarrow$ iii) and iii) $\Rightarrow$ iv) are immediate. We prove the reverse implications in the same order.

Let $\Gamma$ denote the Cayley graph of $G$ with respect to the generating set $X$, and let $\mathcal{T}_{\mathcal{N}}$ be the maximal tree in $\Gamma$ traversed by paths starting at the identity vertex and labeled by words in $\mathcal{N}$.

First, assume ii) holds, and let $\mathcal{S}$ be the stackable system. If $e_{g,x}$ is in the tree $\mathcal{T}_{\mathcal{N}}$ in $\Gamma$, let $\Phi(e_{g,x}) := e_{g,x}$. On the other hand, if $e_{g,x}$ does not lie in the tree $\mathcal{T}_{\mathcal{N}}$ in $\Gamma$ and $\Delta_{g,x}$ is the associated van Kampen diagram in $\mathcal{S}$, let $\Phi(e_{g,x})$ be the path in $\Gamma$ labeled by the word $\phi(e_{g,x})$ in $X^*$ such that $\phi(e_{g,x})x^{-1}$ labels the isolated cell of $\Delta_{g,x}$. Note that this implies that $\phi(e_{g,x})x^{-1}$ is a relator, and thus the possible lengths of the words $\phi(e_{g,x})$ are bounded. Now, we define a map $\nu$ from the set of directed edges in $\Gamma \smallsetminus \mathcal{T}_{\mathcal{N}}$

to $\mathbb{Z}^+$ by $\nu(e_{g,x}) := $ area of $\Delta_{g,x}$. Use the function $\nu$ and the usual ordering on $\mathbb{Z}$ to order the set of recursive edges; this gives a well-founded ordering on $\vec{E}$ such that $e' <_\Phi e$ implies $\nu(e') < \nu(e)$. Thus the group is stackable by Definition 1.1, and i) holds.

Next assume that iii) holds. For each directed edge $e_{g,x}$ of the Cayley graph, let $n(e_{g,x})$ be the smallest area of a fully $\mathcal{N}$-triangular van Kampen diagram with boundary $\mathsf{nf}(g)x\mathsf{nf}(gx)^{-1}$. We define a stackable system $\mathcal{S}$ of fully $\mathcal{N}$-triangular van Kampen diagrams by induction on $n(e_{g,x})$. To start, let $\mathcal{S}$ be the empty set. Note that $n(e_{g,x}) = 0$ if and only if $e_{g,x}$ is in $\mathcal{T}_\mathcal{N}$; in this case add a degenerate diagram $\Delta(e_{g,x})$ to $\mathcal{S}$. Similarly, $n(e_{g,x}) = 1$ if and only if $e_{g,x}$ is not in $\mathcal{T}_\mathcal{N}$ and there is a minimal van Kampen diagram for $\mathsf{nf}(g)x\mathsf{nf}(gx)^{-1}$; place a choice of such a diagram in $\mathcal{S}$. Suppose now that $n(e_{g,x}) > 1$, and we have diagrams $\Delta_{g,x}$ in $(S)$ corresponding to all directed edges with lower value for the function $n$. Let $\Delta'$ be a fully $\mathcal{N}$-triangular van Kampen diagram for $e_{g,x}$ with $n(e_{g,x})$ 2-cells. Using Definition 3.3 iii), $\Delta'$ is the disjoint union of a 2-cell $\sigma$ containing $p_x$ and fully $\mathcal{N}$-triangular diagrams $\Delta_i$, with certain gluings. The isolated edge $e_i$ associated with $\Delta_i$ must satisfy $n(e_i) < n(e_{g,x})$. For each $i$ we replace the subdiagram $\Delta_i$ of $\Delta'$ by the diagram in $\mathcal{S}$ with the same boundary label, to obtain a diagram $\Delta_{g,x}$; add this diagram to $\mathcal{S}$. Then $\mathcal{S}$ is a stackable system, completing the proof of ii).

Finally, assume that iv) holds. Then for each $g \in G$ and $x \in X$, there is a $\mathcal{N}$-labeled van Kampen diagram with boundary label $\mathsf{nf}(g)x\mathsf{nf}(gx)^{-1}$; from among all such diagrams, let $\Delta'_{\mathsf{nf}(g),x}$ be a diagram with the least possible number of 2-cells and let $\alpha(g,x)$ denote this number. We show that for each $g \in G$ and $x \in X$ there is a fully $\mathcal{N}$-triangular van Kampen diagram $\Delta_{g,x}$ with boundary label $\mathsf{nf}(g)x\mathsf{nf}(gx)^{-1}$ by induction on $\alpha(g,x)$.

Suppose that $\alpha(g,x) = 0$. Then the diagram $\Delta'_{g,x}$ is degenerate, and hence fully $\mathcal{N}$-triangular. In this case we can take $\Delta_{g,x} := \Delta'_{g,x}$.

Suppose next that $n := \alpha(g,x) > 0$, and that for all $g' \in G$ and $x' \in X$ with $\alpha(g',x') < n$ there is a fully $\mathcal{N}$-triangular diagram bounded by $\mathsf{nf}(g')x'\mathsf{nf}(g'x')^{-1}$. Let $\Delta' := \Delta'_{\mathsf{nf}(g),x}$ and let $*$ be the basepoint of $\Delta'$.

Suppose that there is a word $w \in \mathcal{N}$ that labels two paths $p, p'$ in $\Delta'$ that start at $*$ and suppose that $p$ and $p'$ can be factored as $p = p_1 p_2 p_3$ and $p' = p_1 p''$ such that $p_2$ is a nonempty edge path whose intersection with the path $p'$ consists exactly of the initial vertex $i(p_2)$ and terminal vertex $t(p_2)$ of $p_2$. Since normal forms in $\mathcal{N}$ label simple paths in the tree $\mathcal{T}_\mathcal{N}$, they must also label simple paths in any van Kampen diagram; hence $i(p_2) \neq t(p_2)$. Now we can factor the path $p' = p_1 p_2' p_3'$ such that $p_2'$ is another path in $\Delta'$ from $i(p_2)$ to $t(p_2)$. Moreover we can write $w = w_1 w_2 w_3 = w_1 w_2' w_3'$ such that for each $i$ the word $w_i$ labels the path $p_i$ and $w_i'$ labels $p_i'$. The images of the paths $p_1 p_2$ and $p_1 p_2'$ under the map $\pi_{\Delta'}$ end at the same vertex in the Cayley graph, and so prefix closure (and uniqueness) of the normal form set imply that $w_2 = w_2'$. A similar argument shows that the path $p_2'$ cannot intersect the path $p$ except at the common endpoints of $p_2$ and $p_2'$. Hence

the path $p_2 p_2'^{-1}$ is a simple loop in $\Delta'$. By the Jordan Curve Theorem, this loop separates the diagram $\Delta'$ into two subsets. We remove the subdiagram of $\Delta'$ contained inside this loop, and glue the two simple paths $p_2$ and $p_2'$; this results in a new van Kampen diagram $\Delta''$ with the same basepoint and boundary. Moreover, the diagram $\Delta''$ is $\mathcal{N}$-labeled, and contains fewer cells than $\Delta'$; this contradicts our choice of $\Delta' = \Delta'_{g,x}$ as a $\mathcal{N}$-labeled triangular van Kampen diagram with minimal number of 2-cells. Hence the set of edges that lie along paths in $\Delta'$ starting at $*$ and labeled by elements of $\mathcal{N}$ must form a maximal tree, since a pair of such paths cannot have prefixes that diverge and then merge.

If the edge $e_{g,x}$ lies in the tree $\mathcal{T}_\mathcal{N}$, then either $\mathsf{nf}(gx) = \mathsf{nf}(g)x$ or $\mathsf{nf}(g) = \mathsf{nf}(gx)x^{-1}$. There is a degenerate, and hence fully $\mathcal{N}$-triangular, van Kampen diagram $\Delta_{g,x}$ $(= \Delta')$ consisting of a line segment labeled $\mathsf{nf}(gx)$ or $\mathsf{nf}(g)$, respectively, in this case.

On the other hand, suppose that $e_{g,x}$ is not in $\mathcal{T}_\mathcal{N}$. Note that $\Delta'$ must contain a 2-cell $\sigma$ with the isolated edge $p_x$ in its boundary, since $p_x$ is the only directed edge in the path along $\partial\Delta'$ mapped by $\pi_{\Delta'}$ to an edge outside of $\mathcal{T}_\mathcal{N}$. If $\Delta'$ contains only one 2-cell, then $\Delta'$ is minimal, and hence fully $\mathcal{N}$-triangular. Suppose that $\Delta'$ contains more than one 2-cell. Let $v_0, v_1, ..., v_t$ be the successive vertices, and $e_1, ...e_t$ the successive edges, of the path in $\partial\sigma \smallsetminus p_x$ from the initial vertex $v_0$ to the terminal vertex $v_t$ of $p_x$. For each $0 \le i \le t$, there is a unique path $p_i$ from the basepoint $*$ to $v_i$ that is labeled by a word in $\mathcal{N}$. For each $0 \le i < t$, the concatenated path $l_i := p_i e_i p_{i+1}^{-1}$ is a loop in $\Delta'$. Let $q_i$ be the maximal common prefix of the pair of paths $p_i, p_{i+1}$; that is, $p_i = q_i r_i$ and $p_{i+1} = q_i s_i$. If $q_i$ equals one of the paths $p_1, p_{i+1}$, then either $q_i = p_i$, $r_i$ is a constant path, and $s_i = e_i$, or $q_i = p_{i+1}$, $r_i = e_i^{-1}$, and $s_i$ is constant; in both cases, the loop $l_i$ follows a line segment in $\Delta'$ and returns along the same segment back to $*$, and we let $\Delta_i$ be the degenerate van Kampen diagram given by this line segment. On the other hand, if $q_i$ is a proper subpath of both $p_i$ and $p_{i+1}$, then the fact that normal forms from $*$ label paths in a tree shows that the path $r_i e_i s_i^{-1}$ is a simple loop in $\Delta'$. Let $\tilde{\Delta}_i$ denote the 2-complex inside this loop (including the bounding loop), and let $\Delta'_i$ be the subdiagram $\Delta'_i := q_i \tilde{\Delta}_i$ of $\Delta'$ with the same basepoint $*$. Again applying the fact that normal forms label paths from $*$ that lie in a tree, for each vertex $v$ of $\Delta'_i$, the path in $\Delta'$ from $*$ to $v$ must lie in $\Delta'_i$; hence $\Delta'_i$ is $\mathcal{N}$-labeled triangular. Since the number of 2-cells in $\Delta'_i$ is at most $n-1$, by our inductive assumption there is a fully $\mathcal{N}$-triangular van Kampen diagram $\Delta_i$ with the same boundary label as $\Delta'_i$.

Now the diagram $\Delta_{g,x}$ built from the disjoint union of the $\Delta_i$ and $\sigma$, glued along the normal form paths $p_i$ and the edges $e_i$, is a fully $\mathcal{N}$-triangular van Kampen diagram with boundary label $\mathsf{nf}(g)x\mathsf{nf}(gx)^{-1}$, and therefore iii) holds.                                                                                        $\square$

## 4. HNN EXTENSIONS AND STACKABILITY

Throughout this section, let $G = H*_\psi$ be the HNN-extension of the group $H$ with isomorphism $\psi : A \to B$ between subgroups $A$ and $B$ and stable letter $s$. Assume that $H$ is finitely generated and let $H = \langle Y \mid R_H \rangle$ be a presentation for $H$, where $Y$ is a finite inverse closed generating set for $H$. Let $X := \{s, s^{-1}\} \cup Y$, and let $\rho : X^* \to G$ be the canonical surjection. Then $G$ has the presentation

$$(1) \qquad G = \langle X \mid R_H, \{sas^{-1} = \psi(a) \mid a \in A\} \rangle.$$

**Notation 4.1.** Let $\mathcal{N}_H$ be a set of normal forms for $H$ over $Y$. Let $\mathcal{N}_{H/A}$ be a subset of $\mathcal{N}_H$ satisfying the properties that the composition $\mathcal{N}_{H/A} \xrightarrow{\rho} H \to H/A$ is a bijection and $1 \in \mathcal{N}_{H/A}$, and similarly let $\mathcal{N}_{H/B} \subseteq \mathcal{N}_H$ be a set of normal forms for a set of coset representatives for $B$ in $H$ that contains 1. The *Britton set of normal forms* [16, p. 181] for $G$ is

$$\mathcal{N}_G := \{h_1 s^{\varepsilon_1} h_2 s^{\varepsilon_2} \dots h_n s^{\varepsilon_n} h \mid n \geq 0, \ h \in \mathcal{N}_H \ \text{ and } \varepsilon_i = \pm 1 \text{ for } 1 \leq i \leq n;$$
$$\text{if } \varepsilon_i = +1 \text{ then } h_i \in \mathcal{N}_{H/B}$$
$$\text{and if } \varepsilon_i = -1 \text{ then } h_i \in \mathcal{N}_{H/A};$$
$$\text{and if } \varepsilon_i = -\varepsilon_{i-1} \text{ then } h_i \neq 1\};$$

that is, $\mathcal{N}_G = (\mathcal{N}_{H/A} s^{-1} \cup \mathcal{N}_{H/B} s)^* \mathcal{N}_H \smallsetminus \cup_{\varepsilon \in \{\pm 1\}} X^* s^\varepsilon s^{-\varepsilon} X^*$. Given such a word $w = h_1 s^{\varepsilon_1} h_2 s^{\varepsilon_2} \dots h_n s^{\varepsilon_n} h$ in $\mathcal{N}_G$, we define $\mathsf{tl}(w) := h_1 s^{\varepsilon_1} h_2 s^{\varepsilon_2} \dots h_n s^{\varepsilon_n}$ and $\mathsf{hd}(w) := h$; we refer to these as the *tail* and *head* of $w$, respectively.

The following is immediate from the definition of Britton's normal forms; we will apply this in the proof of Theorem 4.4 below.

**Lemma 4.2.** *Let* $w = h_1 s^{\varepsilon_1} h_2 s^{\varepsilon_2} \dots h_n s^{\varepsilon_n} h \in \mathcal{N}$ *and let* $\tau \in \mathcal{N}$ *be a tail. If either* $n = 0$, $h_1 \neq 1$, *or* $h_1 = 1$ *and* $s^{\varepsilon_1}$ *equals the last letter of* $\tau$, *we have* $\tau w \in \mathcal{N}$.

Before proceeding to Theorem 4.4, we first need a result about changing generating sets for stackable and autostackable groups.

**Proposition 4.3.** *Let* $H$ *be a stackable group with respect to an inverse-closed generating set* $Y$, *and let* $Z$ *be a finite inverse-closed subset of* $H$. *Then* $H$ *is also stackable with respect to the generating set* $Y \cup Z$. *Moreover, if* $H$ *is autostackable over* $Y$, *then* $H$ *is also autostackable over* $Y \cup Z$.

*Proof.* Let $\Gamma := \Gamma(H, Y)$ and $\Gamma' := \Gamma(H, Y \cup Z)$ be the Cayley graphs for $H$ over $Y$ and $Y \cup Z$, respectively. For each $z \in Z \smallsetminus Y$, fix a word $w_z \in Y^*$ such that $z =_H w_z$.

Let $\Phi$ be a bounded flow function for $H$ over the generating set $Y$, with associated maximal tree $\mathcal{T}$ in $\Gamma$, normal form set $\mathcal{N}$, and stacking function; $\phi = \mathsf{label} \circ \Phi \circ \mathsf{path} : \mathcal{N} \times Y \to Y^*$. The tree $\mathcal{T}$ is also a maximal tree in $\Gamma'$, and the associated normal forms over $Y \cup Z$ is the same set $\mathcal{N}$. Define the function $\phi' : \mathcal{N} \times Y \cup Z \to (Y \cup Z)^*$ by $\phi'(w, y) := \phi(w, y)$ for all $w \in \mathcal{N}$ and $y \in Y$, and $\phi'(w, z) := w_z$ for all $w \in \mathcal{N}$ and $z \in Z \smallsetminus Y$. Then $\phi'$ is a

stacking function for $H$ over $Y \cup Z$, and $\Phi' := \mathsf{path} \circ \phi' \circ \mathsf{label}$ is a bounded flow function for $H$ over $Y \cup Z$.

In the case that $H$ is autostackable over $Y$, and $\mathsf{graph}(\phi)$ is synchronously regular, we have

$$\mathsf{graph}(\phi') = \mathsf{graph}(\phi) \bigcup (\cup_{z \in Z \smallsetminus Y} \mathcal{N} \times \{z\} \times \{w_z\}).$$

Since the class of synchronously regular languages is closed under projection on the first coordinate, finite direct products, and finite unions, then $\mathsf{graph}(\phi')$ is also synchronously regular, and so $H$ is autostackable over $Y \cup Z$. $\qquad\qquad\square$

**Theorem 4.4.** *Let $H$ be a stackable group, let $A, B \leq H$ be finitely generated, and let $\psi : A \to B$ be an isomorphism. Then the HNN extension $G = H*_\psi$ is also stackable.*

*Proof.* Suppose that $H$ is stackable with respect to a normal form set $\mathcal{N}_H$ over an inverse-closed generating set $Y$, and let $\langle Y \mid R_H \rangle$ be the finite stacking presentation associated to the bounded flow function for $H$ over $Y$. Let $\mathcal{N}_{H/A}, \mathcal{N}_{H/B}$ be subsets of $\mathcal{N}_H$ (each containing 1) representing transversals of these subgroups. Using the proof of Proposition 4.3, by possibly extending the flow function and adding relators (of the form $z = w_z$) to the presentation, we may assume that the generating set $Y$ contains a subset $Z_A$ which is an inverse-closed generating set for $A$, as well as the subset $Z_B := \{\psi(a) \mid a \in Z_A\}$ of $H$, which generates $B$. (Note that this does not affect the normal form sets $\mathcal{N}_H$, $\mathcal{N}_{H/A}$ and $\mathcal{N}_{H/B}$.) Using Proposition 3.5, for every word $w \in \mathcal{N}_H$ and $y \in Y$, we have a fully $\mathcal{N}_H$-triangular van Kampen diagram $\Delta_{w,y}^H$ over the stacking presentation for $H$.
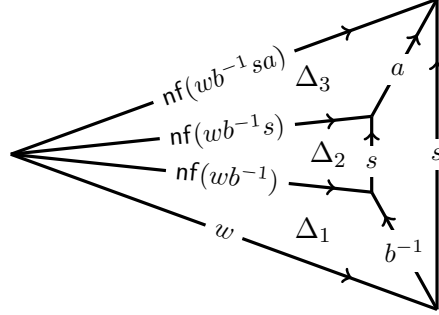
Let $\mathcal{N}_G$ be the Britton normal form set of Notation 4.1. Since the set $\mathcal{N}_H$ is prefix-closed, then the set

$$\mathcal{N}_G = (\mathcal{N}_{H/A}s^{-1} \cup \mathcal{N}_{H/B}s)^* \mathcal{N}_H \cap (X^* \smallsetminus \cup_{\varepsilon \in \{\pm 1\}} X^* s^\varepsilon s^{-\varepsilon} X^*)$$

is also prefix-closed. For all $w \in X^*$, let $\mathsf{nf}(w)$ denote the normal form in $\mathcal{N}_G$ of the element of $G$ represented by $w$. Let $\Gamma := \Gamma(G, X)$ be the Cayley graph for $G$ over $X := Y \cup \{s^{\pm 1}\}$, and let $\mathcal{T}$ be the maximal tree of $\Gamma$ corresponding to the set $\mathcal{N}_G$.

We will show that $G$ is stackable with respect to $\mathcal{N}_G$ by showing that over the finite presentation in Equation (1) there is a fully $\mathcal{N}_G$-triangular van Kampen diagram $\Delta_{w,x}$ with boundary label $wx\mathsf{nf}(wx)^{-1}$ for every normal form word $w \in \mathcal{N}_G$ and generator $x \in X$; that is, we apply Proposition 3.5. We proceed via several cases depending upon $w$ and $x$; in each case, we have that fully $\mathcal{N}_G$-triangular van Kampen diagrams have been constructed for the prior cases.

*Case 1: Suppose that $x \in Y$.* Let $\Delta_{\mathsf{hd}(w),x}^H$ be the fully $\mathcal{N}_H$-triangular van Kampen diagram associated to $\mathsf{hd}(w)$ and $x$. By Lemma 4.2, for all $v \in \mathcal{N}_H$ we have $\mathsf{tl}(w)v \in \mathcal{N}_G$. Thus the diagram $\mathsf{tl}(w)\Delta_{\mathsf{hd}(w),x}^H$ is a fully $\mathcal{N}_G$-triangular van Kampen diagram in $G$, with boundary label $wx\mathsf{nf}(wx)^{-1}$.

FIGURE 1. The diagram $\Delta_{w,s}$.



*Case 2: Suppose that $x = s^{\pm 1}$.* The two cases $x = s$ and $x = s^{-1}$ are analogous, so we assume that $x = s$ and leave the case when $x = s^{-1}$ to the reader.

Let $\mathsf{SL}_B \subset Z_B^*$ denote the set of shortlex normal forms for $B$ over the generating set $Z_B$ with respect to some total ordering of $Z_B$. The element of $H$ represented by $\mathsf{hd}(w)$ can be written $\mathsf{hd}(w) =_H uv$ for a unique $u \in \mathcal{N}_{H/B}$ and $v \in \mathsf{SL}_B$.

We proceed by induction on the length of $v$. If $v = 1$, then either $wx = ws \in \mathcal{N}_G$ or else $w$ ends with $s^{-1}$; in either case, there is a degenerate van Kampen diagram $\Delta_{w,x}$ with boundary word $ws\mathsf{nf}(ws)^{-1}$.

Next suppose that $l(v) > 0$ and that we have a fully $\mathcal{N}_G$-triangular van Kampen diagram for $\mathsf{nf}(\mathsf{tl}(w)uv')s\mathsf{nf}(\mathsf{tl}(w)uv's)^{-1}$ for all $v' \in \mathsf{SL}_B$ with $l(v') < l(v)$. Write $v = v'b$ with $v' \in \mathsf{SL}_B$ and $b \in Z_B$. Let $a \in Z_A$ be the letter satisfying $\psi(a) = b$.

The normal form $\mathsf{nf}(wb^{-1})$ is $w' := \mathsf{tl}(w)\mathsf{nf}(uv')$; then since $\mathsf{hd}(w') =_H uv'$ and $l(v') < l(v)$, our inductive assumption implies that there is a fully $\mathcal{N}_G$-triangular van Kampen diagram $\Delta_2$ with boundary word $\mathsf{nf}(wb^{-1})s\mathsf{nf}(wb^{-1}s)^{-1}$.

Applying Case 1, there are fully $\mathcal{N}_G$-triangular diagrams $\Delta_1 = \mathsf{tl}(w)\Delta_{\mathsf{hd}(w),b^{-1}}^H$ and $\Delta_3 = \mathsf{tl}(wb^{-1}s)\Delta_{\mathsf{hd}(wb^{-1}s),a}^H$ (see Figure 1). Gluing the diagrams $\Delta_1, \Delta_2$ along their (simple) boundary paths $\mathsf{nf}(wb^{-1})$, and gluing the resulting diagram with $\Delta_3$ along their $\mathsf{nf}(wb^{-1}s)$ paths, results in the subdiagram $\tilde{\Delta}$ of Figure 1 with boundary label $wb^{-1}sa$. We glue a single 2-cell with boundary label $b^{-1}sas^{-1}$ to $\tilde{\Delta}$ along their $b^{-1}sa$ boundary paths; this yields a fully $\mathcal{N}_G$-triangular van Kampen diagram $\Delta_{w,x}$ with boundary label $wx\mathsf{nf}(wx)^{-1}$. $\square$

**Remark 4.5.** We record here, for later use, the result of Theorem 4.4 in terms of stacking maps. Suppose that $\phi_H : \mathcal{N}_H \times Y \to Y^*$ is the stacking map associated to the bounded flow function on $H$ in Theorem 4.4. For each word $w \in \mathcal{N}_G$ define $\mathsf{trans}_A(w)$ and $\mathsf{sub}_A(w)$ to be the unique elements of the transversal $\mathcal{N}_{H/A}$ and subgroup shortlex representatives $\mathsf{SL}_A$, respectively, such that $\mathsf{hd}(w) =_H \mathsf{trans}_A(w)\mathsf{sub}_A(w)$, and let $\mathsf{last}(\mathsf{sub}_A(w))$ denote the last letter (in $Z_A$) of the word $\mathsf{sub}_A(w)$. Similarly define $\mathsf{trans}_B(w)$, $\mathsf{sub}_B(w)$, and $\mathsf{last}(\mathsf{sub}_B(w))$. The stacking map for $G$ over $X$ is the function $\phi_G : \mathcal{N}_G \times X \to$

$X^*$ defined for all $w \in \mathcal{N}_G$ and $x \in Y$ by:

$$\phi_G(w,x) := \begin{cases} \phi_H(\mathsf{hd}(w),x) & \text{if } x \in Y \\ x & \text{if } x = s \text{ and } \mathsf{hd}(w) = \mathsf{trans}_B(w) \\ & \text{or } x = s^{-1} \text{ and } \mathsf{hd}(w) = \mathsf{trans}_A(w) \\ \mathsf{last}(\mathsf{sub}_B(w))^{-1}s\psi^{-1}(\mathsf{last}(\mathsf{sub}_B(w))) & \text{if } x = s \text{ and } \mathsf{hd}(w) \neq \mathsf{trans}_B(w) \\ \mathsf{last}(\mathsf{sub}_A(w))^{-1}s^{-1}\psi(\mathsf{last}(\mathsf{sub}_A(w))) & \text{if } x = s^{-1} \text{ and } \mathsf{hd}(w) \neq \mathsf{trans}_A(w). \end{cases}$$

In the case that the graph of the stacking map $\phi_H$ for $H$ is decidable or synchronously regular, the proof of Theorem 4.4 does not show that the same must hold for the graph of the stacking function $\phi_G$. However, in many special cases, this does hold.

**Corollary 4.6.** *Let $H$ be an autostackable [respectively, algorithmically stackable] group over an inverse-closed generating set $Y$. Let $A \leq H$ be generated by a finite inverse-closed set $Z \subseteq Y$ with shortlex normal form set $\mathsf{SL}_A$ (with respect to some total ordering of $Z$), and let $\psi : A \to H$ be a monomorphism with $\psi(Z) \subseteq Y$. Suppose further that there are regular [respectively, decidable] subsets $\mathcal{N}_{H/A}, \mathcal{N}_{H/\psi(A)} \subseteq \mathcal{N}_H$, each containing 1, representing transversals of these subgroups, and that for each $z \in Z$ and $\tilde{z} \in \psi(Z)$, the sets*

$$L_z := \{w \in \mathcal{N}_H \mid w =_H \mathsf{trans}_A(w)\mathsf{sub}_A(w) \text{ for some}$$
$$\mathsf{trans}_A(w) \in \mathcal{N}_{H/A} \text{ and } \mathsf{sub}_A(w) \in \mathsf{SL}_A \cap Z^*z\} \text{ and}$$

$$L'_{\tilde{z}} := \{w \in \mathcal{N}_H \mid w =_H \mathsf{trans}_{\psi(A)}(w)\mathsf{sub}_{\psi(A)}(w) \text{ for some}$$
$$\mathsf{trans}_{\psi(A)}(w) \in \mathcal{N}_{H/\psi(A)} \text{ and } \mathsf{sub}_{\psi(A)}(w) \in \psi(\mathsf{SL}_A) \cap \psi(Z)^*\tilde{z}\}$$

*are also regular [respectively, decidable]. Then the HNN extension $G = H*_\psi$ is autostackable [respectively, algorithmically stackable].*

*Proof.* We give the proof in the autostackable case; the algorithmically stackable proof is similar. Let $s \in G$ be the stable letter of the HNN extension, and let $X = Y \cup \{s^{\pm 1}\}$.

Let $\phi_H$ be the the stacking map for the autostackable structure for $H$, and let $\phi_G$ be the stacking map for $G$ from Remark 4.5. Now $\mathsf{graph}(\phi_H)$ is synchronously regular. Let $\mathsf{proj}_1, \mathsf{proj}_3 : (X^*)^3 \to X^*$ be the projection maps on the first and third coordinates. Note that the normal form set for $H$ is $\mathcal{N}_H = \mathsf{proj}_1(\mathsf{graph}(\phi_H))$, and since synchronously regular languages are closed under projections, the set $\mathcal{N}_H$ is regular. The normal form set for $G$ has the form $\mathcal{N}_G = \mathsf{Tail} \cdot \mathcal{N}_H$ where

$$\mathsf{Tail} := (\mathcal{N}_{H/A}s^{-1} \cup \mathcal{N}_{H/\psi(A)}s)^* \cap (X^* \smallsetminus \cup_{\varepsilon \in \{\pm 1\}} X^* s^\varepsilon s^{-\varepsilon} X^*),$$

and so $\mathcal{N}_G$ is built from regular languages using intersection, union, complementation, concatenation and Kleene star. (See Section 2 for properties of regular and synchronously regular languages.) Hence $\mathcal{N}_G$ also is a regular language.

The graph of $\phi_G$ can be written

$$
\begin{aligned}
\mathsf{graph}(\phi_G) \;=\; & \big( \cup_{y\in Y, v\in\mathsf{proj}_3(\mathsf{graph}(\phi_H))} L_1 \times \{y\} \times \{v\} \big) \\
& \cup \big( \cup_{\varepsilon\in\{\pm 1\}} L_{2,\varepsilon} \times \{s^{\varepsilon}\} \times \{s^{\varepsilon}\} \big) \\
& \cup \big( \cup_{\tilde{z}\in\psi(Z)} L'_{3,\tilde{z}} \times \{s\} \times \{\tilde{z}^{-1} s \psi^{-1}(\tilde{z})\} \big) \\
& \cup \big( \cup_{z\in Z} L_{3,z} \times \{s^{-1}\} \times \{z^{-1} s^{-1} \psi(z)\} \big)
\end{aligned}
$$

where

$$
\begin{aligned}
L_1 &= \mathsf{Tail} \cdot \mathsf{proj}_1(\mathsf{graph}(\phi_H) \cap (X^* \times \{y\} \times \{v\})), \\
L_{2,1} &= \mathsf{Tail} \cdot \mathcal{N}_{H/\psi(A)}, \\
L_{2,-1} &= \mathsf{Tail} \cdot \mathcal{N}_{H/A}, \\
L_{3,z} &= \mathsf{Tail} \cdot L_z, \text{ and} \\
L'_{3,\tilde{z}} &= \mathsf{Tail} \cdot L'_{\tilde{z}}.
\end{aligned}
$$

Again closure properties of regular and synchronously regular languages show that each of these languages is regular, and hence so is $\mathsf{graph}(\phi_G)$. □

In the algorithmically stackable case, Corollary 4.6 can be rephrased in terms of solvability of the subgroup membership problem.

**Corollary 4.7.** *Let $H$ be an algorithmically stackable group, let $A, B \leq H$ be finitely generated, and let $\psi : A \to B$ be an isomorphism. Suppose further that the subgroup membership problem is decidable for the subgroups $A$ and $B$ in $H$. Then the HNN extension $G = H*_\psi$ is also algorithmically stackable.*

*Proof.* From Theorem 4.4 and Remark 4.5, there is a finite inverse-closed generating set $Y$ for $H$ containing a finite inverse-closed set $Z$ of generators for $A$ as well as the generators $\psi(Z)$ of $B = \psi(A)$, and there is a stackable structure for $G = H*_\psi$ over $X = Y \cup \{s^{\pm 1}\}$ with Britton normal form set $\mathcal{N}_G$, such that the associate stacking map is given in Remark 4.5. From Corollary 4.6, then, it suffices to show that there are decidable transversals $\mathcal{N}_{H/A}, \mathcal{N}_{H/B} \subseteq \mathcal{N}_H$, each containing 1, such that the languages $L_z$ and $L'_{\tilde{z}}$ of Corollary 4.6 are also decidable.

Let $<_{\mathsf{SL}}$ denote the shortlex ordering on $Y^*$ corresponding to a total ordering of $Y$. For each coset $hA$ of $H/A$, let $\tau_{hA}$ denote the shortlex least word in $\mathcal{N}_H$ representing an element of $hA$, and let

$$
\mathcal{N}_{H/A} = \{ \tau_{hA} \mid hA \in H/A \}.
$$

Note that the empty word 1 is an element of $\mathcal{N}_{H/A}$.

In order to determine whether a given word $w \in Y^*$ lies in $\mathcal{N}_{H/A}$, first use decidability to determine whether $w \in \mathcal{N}_H$. If not, then (halt and output) $w \notin \mathcal{N}_{H/A}$; if so, we next enumerate the finite set $S$ of elements of $Y^*$ satisfying $v <_{\mathsf{SL}} w$ for all $v \in S$. For each word $v \in S$, use decidability to determine whether $v \in \mathcal{N}_H$ and use the solution of the subgroup membership problem to determine whether $v^{-1}w \in A$. If there is a word $v \in S$ with $v \in \mathcal{N}_H$ and $v^{-1}w \in A$, then $w \notin \mathcal{N}_{H/A}$; and if there is no such word in $S$, then $w \in \mathcal{N}_{H/A}$. Hence $\mathcal{N}_{H/A}$ is decidable.

Next suppose that $z \in Z$ and consider the set of Corollary 4.6

$$L_z := \{u \in \mathcal{N}_H \mid u =_H \tau\sigma \text{ for some } \tau \in \mathcal{N}_{H/A} \text{ and } \sigma \in \mathsf{SL}_A \cap Z^* z\}$$

where (as before) $\mathsf{SL}_A$ is the set of shortlex normal forms for $A$ over $Z$. The algorithm to determine whether a given word $w$ over $Y$ lies in $L_z$ also begins by using decidability to determine whether $w \in \mathcal{N}_H$, and if not, halts with $w \notin L_z$. If $w \in \mathcal{N}_H$, then we repeat the algorithm in the previous paragraph to compute the word $\tau \in \mathcal{N}_{H/A}$ satisfying $\tau = \tau_{wA}$; that is, $\tau A = wA$ and so $\tau^{-1}w \in A$, and moreover $\tau \leq_{\mathsf{SL}} w$ is the shortlex least word with this property. Next enumerate all words $y_0, y_1, y_2, ...$ over $Z$ in increasing shortlex order. Now since the word $\tau^{-1}w$ represents an element of $A$, we have $\tau^{-1}w =_H y_j$ for some indices $j$; we can use the solution of the word problem from the algorithmically stackable structure on $H$ to determine the first index $i$ for which $\tau^{-1}w =_H y_i$. Then $\sigma := y_i \in$
$sll_A$ and $w =_H \tau\sigma$. Now $w \in L_z$ iff $y_i$ ends with the letter $z$. Thus $L_z$ is also decidable.

A similar argument shows that the set

$$\mathcal{N}_{H/B} = \{\tau'_{hB} \mid hB \in H/B\},$$

where $\tau'_{hB}$ denotes the shortlex least word in $\mathcal{N}_H$ representing an element of $hB$, is decidable and contains 1, and for each $\tilde{z} \in \psi(Z)$ the set

$$L'_{\tilde{z}} := \{w \in \mathcal{N}_H \mid w =_H \tau\sigma \text{ for some } \tau \in \mathcal{N}_{H/\psi(A)} \text{ and } \sigma \in \psi(\mathsf{SL}_A) \cap \psi(Z)^* \tilde{z}\}$$

is also decidable. $\qquad\square$

## 5. Applications and Dehn functions

In this section we give three applications of Theorem 4.4 and Corollaries 4.6 and 4.7, that give information on the Dehn functions of stackable, algorithmically stackable, and autostackable groups.

5.1. **Stackable versus autostackable.** In the first application, we show that stackability and autostackability are not the same property, and that the class of stackable groups contains groups whose Dehn function is not computable.

**Theorem 5.1.** *There exists a stackable group with unsolvable word problem, and hence stackability does not imply algorithmic stackability.*

*Proof.* Let $C = \langle Y \mid R \rangle$ be a finitely presented group with unsolvable word problem. Let $Y'$ be a copy of $Y$, and let $H = F(Y) \times F(Y')$ be a direct copy of the free groups generated by $Y$ and $Y'$. Also let $\rho : F(Y) \to C$ and $\rho' : F(Y') \to C$ be the quotient maps. Let $A$ be the *Mihailova subgroup*

$$A = \{(h, h') \in H \mid \rho(h) = \rho'(h')\}$$

associated to $C$. Mihailova [17] showed that the subgroup membership problem for $A$ in $H$ is not decidable; that is, there does not exist an algorithm

that upon input of a word $w$ in the generating set $(Y \cup Y')^{\pm 1}$ of $H$, can determine whether $w$ represents an element of the subgroup $A$. The group $A$ is finitely generated (see for example the paper of Bogopolski and Ventura [4] for a discussion and recursive presentation for this group); let $Z$ be a finite generating set for $A$.

To construct an HNN extension from this data, we let $\psi : A \to A$ be the identity function on $A$, and let $G = H *_\psi$. Let $\widetilde{Y} := Y \cup Y' \cup Z$, and for each $z \in Z$, let $w_z \in ((Y \cup Y')^{\pm 1})^*$ be a word satisfying $z =_H w_z$. Then

$$G = \langle \widetilde{Y} \cup \{s\} \mid \quad [y, y'] = 1 \text{ for all } y \in Y, y' \in Y', \text{ and}$$
$$z = w_z \text{ and } szs^{-1} = z \text{ for all } z \in Z \rangle.$$

Since the group $H$ is a direct product of free groups, $H$ is a stackable group. Now Theorem 4.4 and Remark 4.5 show that $G$ is a stackable group, with a stackable structure over the generating set $(\widetilde{Y} \cup \{s\})^{\pm 1}$ yielding the above as the stacking presentation.

If the word problem for $G$ were to have a solution, then upon input of any word $sws^{-1}w^{-1}$ with $w \in (Y \cup Y')^{\pm 1 *} \subseteq \widetilde{Y}^{\pm 1 *}$, the word problem algorithm can determine whether or not $sws^{-1}w^{-1} =_G \varepsilon$. However, $sws^{-1}w^{-1} =_G \varepsilon$ if and only if $w$ represents an element of the subgroup $A$ in the domain of $\psi$. Hence this solves subgroup membership as well, giving a contradiction. Since $G$ does not have solvable word problem, this stackable group $G$ cannot be algorithmically stackable. $\qquad \square$

5.2. **Dehn functions for algorithmic stackability: Hydra groups.** In our second application, we show that Dehn functions of algorithmically stackable groups can be extremely large.

**Theorem 5.2.** *The class of algorithmically stackable groups includes groups with Dehn functions in each level of the Grzegorczyk hierarchy of primitive recursive functions.*

*Proof.* Dison and Riley [12] defined a family of groups $\Gamma_k$ (for $k \geq 2$), built by HNN extensions, and showed that the Dehn function of $\Gamma_k$ is equivalent to the $k$-th Ackerman function. In particular, for each integer $k \geq 2$, the group $\Gamma_k = G_k *_{\psi_k}$ is an HNN extension of a free-by-cyclic group

$$G_k = \langle a_1, ..., a_k, t \mid ta_1 t^{-1} = a_1, \ ta_i t^{-1} = a_i a_{i-1} \ (i > 1) \rangle$$

(known as a hydra group) with respect to the identity map $\psi_K : H_k \to H_k$ on the finitely generated (rank $k$ free) subgroup $H_k = \langle a_1 t^{-1}, ..., a_k t^{-1} \rangle$. Since the class of algorithmically stackable groups is closed under extension [8], the group $G_k$ is algorithmically stackable. Theorem 4.4 shows that $\Gamma_k$ is also stackable. Dison, Einstein and Riley [11, Theorem 3] have shown that for the subgroup $H_k$ of $G_k$, the subgroup membership problem is decidable. Then Corollary 4.7 shows that $\Gamma_k$ is also algorithmically stackable. $\qquad \square$

5.3. **Dehn functions for autostackability: The Baumslag-Gersten group.** In this third application we consider the Baumslag-Gersten group, also known as Baumslag's (nonmetabelian) group, which is presented by

$$G = \langle a, s \mid (sas^{-1})a(sa^{-1}s^{-1}) = a^2 \rangle = \langle a, t, s \mid tat^{-1} = a^2, \ sas^{-1} = t \rangle.$$

This group can be realized as an HNN extension $G = H*_\psi$ where $H$ is the Baumslag-Solitar group $H = BS(1,2) = \langle a, t \mid tat^{-1} = a^2 \rangle$ and $\psi : \langle a \rangle \to \langle t \rangle$ is the map given by $\psi(a) = t$. The group $H$ is autostackable [6, 7], and so Theorem 4.4 shows that $G$ is stackable. We strengthen this result to show the following.

**Theorem 5.3.** *The Baumslag-Gersten group $\langle a, s \mid (sas^{-1})a(sa^{-1}s^{-1}) = a^2 \rangle$ is autostackable.*

*Proof.* The Baumslag-Solitar group $H = \langle a, t \mid tat^{-1} = a^2 \rangle$ has a finite complete rewriting system on the generating set $Y = \{a^{\pm 1}, t^{\pm 1}\}$ given by

$$\{a^\varepsilon a^{-\varepsilon} \to 1, \ t^\varepsilon t^{-\varepsilon} \to 1, \ a^2 t \to ta, a^{-1}t \to ata^{-1}, \ a^\varepsilon t^{-1} \to t^{-1}a^{2\varepsilon} \mid \varepsilon \in \{\pm 1\} \},$$

and hence is autostackable [7]. The normal form set of this autostackable structure is the regular language

$$\begin{aligned}
\mathcal{N}_H &= [\{t, at, t^{-1}\}^* \smallsetminus \cup_{\varepsilon \in \{\pm 1\}} Y^* t^\varepsilon t^{-\varepsilon} Y^*](a^* \cup (a^{-1})^*) \\
&= [(t^{-1})^* \cup ((t^{-1})^* at \cup 1)(\{1, a\} \cdot t)^*](a^* \cup (a^{-1})^*).
\end{aligned}$$

(That is, $H$ is an HNN extension of the infinite cyclic group $\langle a \rangle$ by the monomorphism $\langle a \rangle \to \langle a \rangle$ defined by $a \mapsto a^2$, and $\mathcal{N}_H$ is the associated set of Britton normal forms.)

Let $A = \langle a \rangle$ and $B = \langle t \rangle$, subgroups of $H$, and let $\psi : A \to B$ be the map $\psi(a) = t$, so that $G = H*_\psi$. Then the generating set $Z := \{a^{\pm 1}\}$ for $A$ and its image $\psi(Z) = \{t^{\pm 1}\}$ are both subsets of the generating set $Y$ of $H$.

By Corollary 4.6, it now suffices to show that there are regular transversals for these subgroups such that each of the languages $L_z$ and $L'_{\tilde z}$ is regular.

Define

$$\begin{aligned}
\mathcal{N}_{H/A} &:= (t^{-1})^* \cup ((t^{-1})^* at \cup 1)(\{1, a\} \cdot t)^* \quad \text{and} \\
\mathcal{N}_{H/B} &:= a^* \cup (a^{-1})^* \cup t^{-1}(t^{-1})^* \cdot (a(a^2)^* \cup a^{-1}(a^{-2})^*).
\end{aligned}$$

Then $\mathcal{N}_{H/A}$ and $\mathcal{N}_{H/B}$ are subsets of $\mathcal{N}_H$ (each containing 1) that are transversals for $A$ and $B$ in $H$, respectively. We note that $\mathcal{N}_{H/A}$ and $\mathcal{N}_{H/B}$ are built from finite sets using unions, concatenations, and Kleene star, and so both of these sets are also regular languages.

The set of shortlex normal forms for elements of the subgroup $A$ over the generating set $Z$ is $\mathsf{SL}_A = a^* \cup (a^{-1})^*$, and similarly the shortlext normal forms for $B$ over $\psi(Z)$ is $\mathsf{SL}_B = t^* \cup (t^{-1})^*$.

Let $z \in Z$. Then $z = a^\varepsilon$ for some $\varepsilon \in \{\pm 1\}$. Since $\mathcal{N}_H = \mathcal{N}_{H/A}\mathsf{SL}_A$, the language $L_{a^\varepsilon}$ satisfies

$$
\begin{aligned}
L_{a^\varepsilon} &= \{w \in \mathcal{N}_H \mid w =_H \mathsf{trans}_A(w)\mathsf{sub}_A(w) \text{ for some } \mathsf{trans}_A(w) \in \mathcal{N}_{H/A} \\
&\qquad \text{and } \mathsf{sub}_A(w) = a^{\varepsilon i} \text{ with } i > 0\} \\
&= \mathcal{N}_H \cap Y^* a^\varepsilon.
\end{aligned}
$$

Then $L_{a^\varepsilon}$ is an intersection of regular languages, and hence is also regular.

Next, for $\tilde{z} \in \psi(Z)$, we have $\tilde{z} = t^\varepsilon$ with $\varepsilon \in \{\pm 1\}$. Suppose first that $\varepsilon = 1$. Then

$$
\begin{aligned}
L_t' &= \{w \in \mathcal{N}_H \mid w =_H \mathsf{trans}_B(w)\mathsf{sub}_B(w) \text{ for some } \mathsf{trans}_B(w) \in \mathcal{N}_{H/\psi(A)} \\
&\qquad \text{and } \mathsf{sub}(w) = t^k \text{ with } k > 0\}.
\end{aligned}
$$

Suppose that $w \in \mathcal{N}_H \cap Y^*tY^*$; that is, $w \in ((t^{-1})^*at \cup t)(\{1, a\} \cdot t)^*](a^* \cup (a^{-1})^*)$. Then either $w = t^{-i}ata^{\varepsilon_1} \cdots ta^{\varepsilon_k}ta^\ell$, or $w = ta^{\varepsilon_1} \cdots ta^{\varepsilon_k}ta^\ell$ for some $i, k \geq 0$, $\varepsilon_i \in \{0, 1\}$ and $\ell \in \mathbb{Z}$. Hence either $\mathsf{trans}_B(w) = t^{-i}a^{1+2\varepsilon_1+\cdots+2^k\varepsilon_k+2^{k+1}\ell}$ or $\mathsf{trans}_B(w) = a^{2\varepsilon_1+\cdots+2^k\varepsilon_k+2^{k+1}\ell}$ (respectively), and $\mathsf{sub}_B(w) = t^{k+1}$. Since $k \geq 0$, then $\mathsf{last}(\mathsf{sub}(w)) = t$, and so $w \in L_t'$. Hence $L_t' \supseteq \mathcal{N}_H \cap Y^*tY^*$. On the other hand, for any $v \in L_t'$, we have $v = \mathsf{nf}(a^jt^k)$ or $v = \mathsf{nf}(t^{-i}a^{2j+1}t^k)$ for some $i \geq 0$, $j \in \mathbb{Z}$, and $k > 0$. Applying the rules of the rewriting system above, then the normal form $v$ must contain the letter $t$. That is,

$$L_t' = \mathcal{N}_H \cap Y^*tY^*$$

and therefore this set is a regular language.

Finally we consider the set

$$
\begin{aligned}
L_{t^{-1}}' &= \{w \in \mathcal{N}_H \mid w =_H \mathsf{trans}_B(w)\mathsf{sub}_B(w) \text{ for some } \mathsf{trans}_B(w) \in \mathcal{N}_{H/\psi(A)} \\
&\qquad \text{and } \mathsf{sub}(w) = t^{-i} \text{ with } i > 0\}.
\end{aligned}
$$

In this case we have

$$L_{t^{-1}}' = \mathcal{N}_H \smallsetminus (\mathcal{N}_{H/B} \cup L_t') = t^{-1}(t^{-1})^*((a^2)^* \cup (a^{-2})^*),$$

and so $L_{t^{-1}}'$ is also regular.

Corollary 4.6 now shows that $G$ is autostackable. $\qquad\square$

The following Corollary is now immediate from Theorem 5.3 and Platonov's proof that the Dehn function of the Baumslag-Gersten group is not elementary [18].

**Corollary 5.4.** *The class of autostackable groups includes groups with nonelementary Dehn functions.*

## 6. Autostackable metabelian groups

In this section we consider an infinite family of nonconstructible metabelian groups. Let $p \in \{n \in \mathbb{Z} \mid n \geq 2\} \cup \{\infty\}$, and let

$$G_p = \langle a, s, t \mid a^p = 1, [a^t, a] = 1, a^s = a^ta, [s, t] = 1 \rangle,$$

where the case $p = \infty$ means that no relation $a^p = 1$ occurs. The group $G_\infty$ is Baumslag's metabelian group, which is introduced in [3], and for $p < \infty$,

the torsion analog $G_p$ of Baumslag's metabelian group is the Diestel-Leader group $\Gamma_3(p)$ (which is also metabelian). Our objective in this section is to show in Theorem 6.1 that $G_\infty$ is algorithmically stackable and the groups $G_p$ for $p < \infty$ are autostackable.

We begin with a description of the subgroup structure of $G_p$, following [3]. Let $H_p$ be the subgroup of $G_p$ generated by $Y = \{a^{\pm 1}, t^{\pm 1}\}$. In his paper [3], Baumslag showed that a consequence of the relations in the presentation above of $G_\infty$ is that $[a^{t^i}, a^{t^j}] = 1$ for all $i, j \in \mathbb{Z}$. Moreover

$$
\begin{aligned}
H_p &= \langle a, t \mid [a^{t^i}, a^{t^j}] = 1 \text{ for all } i, j \in \mathbb{Z}, a^p = 1 \rangle \\
&= \Big( \bigoplus_{i \in \mathbb{Z}} \langle t^i a t^{-i} \rangle \Big) \rtimes \langle t \rangle,
\end{aligned}
$$

where $\langle t^i a t^{-i} \rangle$ is isomorphic to $\mathbb{Z}_p$ for each $i$, $\langle t \rangle \cong \mathbb{Z}$, and $t$ acts on $\bigoplus_{i \in \mathbb{Z}} \langle t^i a t^{-i} \rangle$ conjugating the $i$-th summand to the $(i+1)$-th summand; that is, $H_p$ is the (restricted) wreath product $H_p = \mathbb{Z}_p \wr \mathbb{Z}$. (In the case that $p = 2$, the group $H_2$ is also known as the lamplighter group.) Let $\psi : H_p \to \langle a^t a, t \rangle \le H_p$ be the map defined by $\psi(a) = a^t a$ and $\psi(t) = t$; then the group $G_p$ is the HNN extension $G_p = H_p *_{\psi_p}$, and the generator $s$ of $G_p$ is the corresponding stable letter.

The crucial difference with Theorem 4.4 is that in this case the group $H_p$ is not finitely presentable, so $H_p$ cannot have a stackable structure. Despite this, there are some analogies between the proofs of Theorems 4.4 and 6.1; in particular, the Britton set of normal forms are used for the HNN extensions in both.

In order to describe the normal form set for $G_p$, and to streamline other parts of the proof of Theorem 6.1, we also make use of an another way to view the elements of this group. Using the isomorphism $\hat{\rho}$ between $\big( \bigoplus_{i \in \mathbb{Z}} \langle t^i a t^{-i} \rangle \big)$ and $\mathbb{Z}_p[x, \frac{1}{x}]$ given by $\hat{\rho}((t^{i_1} a^{\beta_1} t^{-i_1}) \cdots (t^{i_n} a^{\beta_n} t^{-i_n})) := \beta_1 x^{i_1} + \cdots + \beta_n x^{i_n}$, there are isomorphisms

$$
\hat{\rho} : H_p \;\to\; \widehat{H}_p := \mathbb{Z}_p[x, \frac{1}{x}] \rtimes \langle \hat{t} \rangle \qquad \text{and}
$$

$$
\hat{\rho} : G_p \;\to\; \widehat{G}_p := \big( \mathbb{Z}_p[x, \frac{1}{x}] \rtimes \langle \hat{t} \rangle \big) *_{\hat{\psi}}
$$

(with $\hat{\rho}(t) := \hat{t}$ and $\hat{\rho}(s) := \hat{s}$) where the conjugation action of $\hat{t}$ on $\mathbb{Z}_p[x, \frac{1}{x}]$ is multiplication by $x$, and the map $\hat{\psi} : H_p \to \langle 1 + x \rangle \rtimes \langle \hat{t} \rangle$ is defined by $\hat{\psi}(x^0) := 1 + x$ and $\hat{\psi}(\hat{t}) := \hat{t}$; that is, the conjugation action by the stable letter $\hat{s}$ on $\widehat{H}_p$ is given by multiplication by $1 + x$ on the $\mathbb{Z}_p[x, \frac{1}{x}]$ subgroup and fixes $\hat{t}$.

**Theorem 6.1.** *Baumslag's metabelian group $G_\infty = \langle a, s, t \mid a^s = a^t a, [a^t, a] = 1, [s, t] = 1 \rangle$ is algorithmically stackable, and the Diestel-Leader torsion analogs $G_p = \langle a, s, t \mid a^s = a^t a, [a^t, a] = 1, [s, t] = 1, a^p = 1 \rangle$ with $p \in \mathbb{Z}$ and $p \ge 2$ are autostackable.*

*Proof.* Although we cannot directly apply Theorem 4.4, we will use roughly the same ingredients in order to build a stacking system for the group $G_p$. Consider the inverse closed generating set $X = \{a^{\pm 1}, s^{\pm 1}, t^{\pm 1}\}$ for $G_p$.

*Step I. Normal forms and notation:*

In order to build a set of normal forms for the group $H_p$, we use the isomorphism $H_p \cong \widehat{H}_p$, and note that an arbitrary element of $\widehat{H}_p$ can be written uniquely in the form $q(x)\hat{t}^m$ where $q(x) \in \mathbb{Z}_p[x, \frac{1}{x}]$ and $m \in \mathbb{Z}$. Let

$$\mathcal{N}_{H_p} := \{t^m \mid m \in \mathbb{Z}\} \cup$$
$$\{t^r a^{\alpha_r} t a^{\alpha_{r+1}} \ldots t a^{\alpha_l} t^{-l+m} \mid r, l, m \in \mathbb{Z}, r \leq l, \alpha_i \in \mathbb{Z}_p \text{ for } r \leq i \leq l,$$
$$\text{and } \alpha_r, \alpha_l \neq 0\};$$

here (and throughout this proof) we write $\alpha_i \in \mathbb{Z}_p$ to mean that $\alpha_i \in \mathbb{Z}$ in the case that $p = \infty$, and $\alpha_i \in \{0, 1, ..., p-1\}$ if $p$ is finite. Then the restriction of the map $\hat{\rho}$ to the set $\mathcal{N}_{H_p}$ gives a bijection $\rho : \mathcal{N}_{H_p} \to \widehat{H}_p$ defined by $\rho(t^m) := \hat{t}^m$ and

$$\rho(t^r a^{\alpha_r} t a^{\alpha_{r+1}} \ldots t a^{\alpha_l} t^{-l+m}) := (\alpha_r x^r + \alpha_{r+1} x^{r+1} + \cdots + \alpha_l x^l)\hat{t}^m,$$

where the integers $r$ and $l$ are the lowest and highest degrees in the polynomial in the $\mathbb{Z}_p[x, \frac{1}{x}]$ subgroup, respectively, and hence $\mathcal{N}_{H_p}$ is a set of normal forms for $H_p$.

The HNN extension $G_p = H_p*_\psi$ is strictly ascending, in that the isomorphism $\psi : A \to B$ of subgroups of $H_p$ maps the full group $A = H_p$ to the proper subgroup $B = \langle a^t a, t \rangle$. The set $\mathcal{N}_{H/A} := \{1\} \subseteq \mathcal{N}_{H_p}$ is a transversal for $H/A$. Under the map $\hat{\rho}$ the subgroup $B$ is isomorphic to the split extension by $\mathbb{Z} = \langle \hat{t} \rangle$ of the ideal $I$ of $\mathbb{Z}_p[x, 1/x]$ generated by $1 + x$. This implies that $H/B \cong \mathbb{Z}_p[x, 1/x]/I \cong \mathbb{Z}_p$, and the set

$$\mathcal{N}_{H/B} := \{a^\beta \mid \beta \in \mathbb{Z}_p\}$$

is a set of normal forms of a set of representatives of the cosets of $B$ in $H$. The corresponding Britton normal form set for the HNN extension $G_p$ is given by

$$\mathcal{N}_{G_p} := \{s^{-k} a^{\beta_1} s a^{\beta_2} s \ldots s a^{\beta_n} s h \mid k, n \geq 0, \beta_i \in \mathbb{Z}_p \text{ for } 1 \leq i \leq n,$$
$$\beta_1 \neq 0 \text{ if both } k > 0 \text{ and } n > 0, \text{ and } h \in \mathcal{N}_{H_p}\}.$$

Given such a word $u = s^{-k} a^{\beta_1} s a^{\beta_2} s \ldots s a^{\beta_n} s h \in \mathcal{N}_{G_p}$, as in Notation 4.1 we denote the *tail* and *head* of $u$ as $\mathsf{tl}(u) := s^{-k} a^{\beta_1} s a^{\beta_2} s \ldots s a^{\beta_n} s$ and $\mathsf{hd}(u) := h$, respectively. Moreover, let $q_u(x) \in \mathbb{Z}_p[x, \frac{1}{x}]$ and $m_u \in \mathbb{Z}$ be defined by $q_u = 0$ and $\rho(\mathsf{hd}(u)) = \hat{t}^{m_u}$ in the case that $\mathsf{hd}(u)$ is a power of $t$, and $\rho(\mathsf{hd}(u)) = q_u(x)\hat{t}^{m_u}$ otherwise. Also in the latter case let $r_u$ and $l_u$ denote the lowest and highest degrees, respectively, of monomials in $q_u(x)$. Let $\alpha_{r_u,u}, ..., \alpha_{l_u,u}$ (or $\alpha_{r_u}, ..., \alpha_{l_u}$ when there is no ambiguity) denote the respective coefficients in the Laurent polynomial $q_u$. In the case that $q_u \neq 0$, note that $\alpha_{l_u,u} \neq 0$; in the remainder of this proof, $\alpha_{l_u,u} = 0$ implies the opposite case that $q_u = 0$.

For all $w \in X^*$, let $\mathsf{nf}(w)$ denote the normal form in $\mathcal{N}_{G_p}$ of the element of $G_p$ represented by $w$. We note that the language $\mathcal{N}_{G_p}$ is prefix-closed. Let $\Gamma := \Gamma(G_p, X)$ be the Cayley graph for $G_p$ over $X$, let $\vec{E}$ and $\vec{P}$ be the sets of directed edges and directed paths in $\Gamma$, and let $\mathcal{T}$ be the maximal tree of $\Gamma$ corresponding to the set $\mathcal{N}_{G_p}$. For all $u \in \mathcal{N}_{G_p}$ and $z \in X$, let $e_{u,z}$ denote the directed edge in $\Gamma$ labeled by $z$ with initial vertex labeled by the element of $G_p$ represented by $u$.

*Step II. The stackable system of fully $\mathcal{N}_{G_p}$-triangular van Kampen diagrams for $p < \infty$:*

In this step we obtain the stackable structure in the case that $p$ is finite by applying Proposition 3.5 and showing that over the finite presentation

$$G_p \quad = \quad \langle a, s, t \mid a^p = 1, [a^t, a] = 1, [s, t] = 1, sa^\alpha s^{-1} = ta^\alpha t^{-1}a^\alpha,$$
$$sa^\alpha ts^{-1} = a^\alpha ta^\alpha \ (\alpha \in \{1, ..., p-1\}) \ \rangle,$$

there is a fully $\mathcal{N}_{G_p}$-triangular van Kampen diagram $\Delta_{u,z}$ with boundary label $uz\mathsf{nf}(uz)^{-1}$ for every normal form word $u \in \mathcal{N}_{G_p}$ and generator $z \in X$. We proceed via several cases depending upon $u$ and $z$; in each case, we have that fully $\mathcal{N}_{G_p}$-triangular van Kampen diagrams have been constructed for the prior cases. Also in each case we record the corresponding function $\Phi : \vec{E} \to \vec{P}$ on the edge $e_{u,z}$, and the algorithm to compute this function.

**Case 1**: *Suppose that $z = t^{\pm 1}$.* In this case, either the word $ut^{\pm 1}$ is in normal form, or else the word $u$ ends with the letter $t^{\pm 1}$. Thus there is a degenerate (and hence fully $\mathcal{N}_{G_p}$-triangular) van Kampen diagram $\Delta_{u,z}$ with boundary label $uz\mathsf{nf}(uz)^{-1}$, and $\Phi(e_{u,z}) := e_{u,z}$.

**Case 2**: *Suppose that $z = a^{\pm 1}$ and either $q_u(x) = 0$ or both $q_u(x) \neq 0$ and $m_u - l_u \geq 0$.*
**Case 2.1**: *Suppose further that $q_u(x) \neq 0$ and $m_u - l_u = 0$.* Then the word $u$ ends with the suffix $a^{\alpha_{l_u}}$ with $\alpha_{l_u} > 0$, and the last letter of $u$ is $a$. If either $z = a^{-1}$ or else both $z = a$ and $\alpha_{l_u} < p - 1$, then again there is a degenerate diagram $\Delta_{u,z}$ and $\Phi(e_{u,z}) := e_{u,z}$. On the other hand, if $z = a$ and $\alpha_{l_u} = p - 1$, then we can factor $u = u'a^{p-1}$ for some $u' \in \mathcal{N}_{G_p}$, and so there is a minimal diagram $\Delta_{u,z} = u'\Delta_{a^{p-1},a}$ with a single 2-cell $\Delta_{a^{p-1},a}$ with boundary label $a^p$; in this case, $\Phi(e_{u,z}) := \mathsf{path}(u, a^{-(p-1)})$.

**Case 2.2**: *Suppose further that either $q_u(x) = 0$ or both $q_u(x) \neq 0$ and $m_u - l_u > 0$.* That is, either $\mathsf{hd}(u) = t^{m_u}$, or $\mathsf{hd}(u)$ contains the letter $a$ and ends with the letter $t$. If $z = a$ then $uz \in \mathcal{N}_{G_p}$ and there is a degenerate diagram $\Delta_{u,z}$; thus $\Phi(e_{u,z}) := e_{u,z}$. If $z = a^{-1}$, then $\mathsf{nf}(uz) = ua^{p-1}$ and there is a minimal diagram $\Delta_{u,z} = u\Delta_{1,a^{-1}}$ with a single 2-cell $\Delta_{1,a^{-1}}$ with boundary label $a^p$; in this case, $\Phi(e_{u,z}) := \mathsf{path}(u, a^{p-1})$.

**Case 3**: *Suppose that $z = a^{\pm 1}$, $q_u(x) \neq 0$, and $m_u - l_u = -1$.* Define $\delta \in \{\pm 1\}$ by $z = a^\delta$. We begin the construction of the fully $\mathcal{N}_{G_p}$-triangular van Kampen diagram $\Delta_{u,z}$ with an isolated cell with one edge labeled $z$, and the remaining boundary path in the other direction labeled $ta^{-1}t^{-1}a^\delta tat^{-1}$;

that is, we take $\Phi(e_{u,z}) := \mathsf{path}(u, ta^{-1}t^{-1}a^\delta tat^{-1})$. By case 1, we already have fully $\mathcal{N}_{G_p}$-triangular diagrams $\Delta_{u,t}$, $\Delta_{\mathsf{nf}(uta^{-1}),t^{-1}}$, $\Delta_{\mathsf{nf}(uta^{-1}t^{-1}a^\delta),t}$, and $\Delta_{\mathsf{nf}(uta^{-1}t^{-1}a^\delta ta),t^{-1}}$. Hence in order to show that we can complete this to a fully $\mathcal{N}_{G_p}$-triangular diagram, it suffices to check that fully $\mathcal{N}_{G_p}$-triangular diagrams have already been constructed for the pairs $(u_1, a^{-1})$, $(u_2, a^\delta)$, and $(u_3, a)$ where $u_1 = \mathsf{nf}(ut)$, $u_2 = \mathsf{nf}(uta^{-1}t^{-1})$, and $u_3 = \mathsf{nf}(uta^{-1}t^{-1}a^\delta t)$.

Note that the word $u$ ends with the suffix $at^{-1}$. Then $u_1$ is the prefix of $u$ with the last letter $t^{-1}$ removed, and so $u_1$ ends with the letter $a$. Hence $\Delta_{u_1,a^{-1}}$ was built in case 2.1.

We prove that $\Delta_{u_2,a^\delta}$ has already been constructed by induction on $\alpha_{l_u,u}$ (and a prior case). The normal form $u_2$ is obtained from $u$ by removing the suffix $at^{-1}$, appending $t^{-1}$, and possibly free reduction (in the case that $\alpha_{l_u,u} = 1$ and either $l_u > r_u$ or $l_u = r_u > 0$); then either ($\mathsf{hd}(u_2) = t^{r_u-1}$ and $\alpha_{l_{u_2},u_2} = 0$), or ($\mathsf{hd}(u_2)$ contains the letter $a$ and $m_{u_2} - l_{u_2} = 0$), or ($\mathsf{hd}(u_2)$ contains the letter $a$, $m_{u_2} - l_{u_2} = -1$ and $\alpha_{l_{u_2},u_2} < \alpha_{l_u,u}$). Hence the construction of $\Delta_{u_2,a^\delta}$ follows from case 2 or induction.

Writing $u = \mathsf{tl}(u)t^{r_u}a^{\alpha_{r_u,u}}\cdots ta^{\alpha_{l_u,u}}t^{-1}$, we have

$$
\begin{aligned}
u_3 \;=_{G_p}\; & \mathsf{tl}(u)t^{r_u}a^{\alpha_{r_u,u}}\cdots ta^{\alpha_{l_u,u}-1}t^{-1}a^\delta t \\
=_{G_p}\; & \mathsf{tl}(u)(t^{r_u}a^{\alpha_{r_u,u}}t^{-r_u})\cdots(t^{l_u}a^{\alpha_{l_u,u}-1}t^{-l_u})(t^{l_u-1}a^\delta t^{-(l_u-1)})t^{l_u} \\
=_{G_p}\; & \mathsf{tl}(u)(t^{r_u}a^{\alpha_{r_u,u}}t^{-r_u})\cdots(t^{l_u-1}a^{\alpha_{l_u-1,u}+\delta}t^{-(l_u-1)})(t^{l_u}a^{\alpha_{l_u,u}-1}t^{-l_u})t^{l_u}.
\end{aligned}
$$

Using the polynomial viewpoint, $q_{u_3}(x) = q_u(x) + \delta x^{l_u-1} - x^{l_u}$. Now either $\mathsf{hd}(u_3) = t^{m_{u_3}}$ (if $q_{u_3} = 0$), or $m_{u_3} - l_{u_3} \geq 0$ (otherwise); hence $\Delta_{u_3,a}$ was also constructed in case 2.

**Case 4**: *Suppose that $z = s^{-1}$ and $\mathsf{tl}(u)$ does not have a suffix of the form $as$.* Either $\mathsf{tl}(u) = s^{-k}$ or $\mathsf{tl}(u) = s$ or $\mathsf{tl}(u)$ ends with $s^2$; the property that we exploit in this case is that for all of these options, we have $\mathsf{tl}(\mathsf{nf}(\mathsf{tl}(u)s^{-1}h)) = \mathsf{nf}(\mathsf{tl}(u)s^{-1})$ and $\mathsf{hd}(\mathsf{nf}(\mathsf{tl}(u)s^{-1}h)) = \mathsf{nf}(h)$ for all $h \in \{a^{\pm 1}, t^{\pm 1}\}^*$. We proceed by induction on the length $l(\mathsf{hd}(u))$ of the head of $u$.

Suppose first that $l(\mathsf{hd}(u)) = 0$. Since $u = \mathsf{tl}(u)$, either $u = s^{-k}$ (with $k \leq 0$), $u = s$, or $u$ ends with $s^2$. In all three of these options, there is a degenerate diagram $\Delta_{u,z}$, and we set $\Phi(e_{u,z}) := e_{u,z}$.

Now suppose that $l(\mathsf{hd}(u)) > 0$, and write $u = u'z'$ where $z' \in \{a, t^{\pm 1}\}$.

If $z' = t^\delta$ with $\delta \in \{\pm 1\}$, then we begin the construction of the fully $\mathcal{N}_{G_p}$-triangular van Kampen diagram $\Delta_{u,z}$ with an isolated cell with one edge labeled $z$, and the remaining boundary path in the other direction labeled $t^{-\delta}s^{-1}t^\delta$, and hence set $\Phi(e_{u,z}) := \mathsf{path}(u, t^{-\delta}s^{-1}t^\delta)$. Now fully $\mathcal{N}_{G_p}$-triangular van Kampen diagrams $\Delta_{u,t^{-\delta}}$ and $\Delta_{\mathsf{nf}(ut^{-\delta}s^{-1}),t^\delta}$ are constructed in case 1, and since $\mathsf{nf}(ut^{-\delta}) = u'$ satisfies $l(u') = l(u) - 1$, the diagram $\Delta_{\mathsf{nf}(ut^{-\delta}),s^{-1}}$ has been built by induction.

On the other hand if $z' = a$, we build $\Delta_{u,z}$ starting with an isolated cell with one edge labeled $z$, and the remaining boundary path in the other direction labeled $a^{-1}s^{-1}atat^{-1}$, and so $\Phi(e_{u,z}) := \mathsf{path}(u, a^{-1}s^{-1}atat^{-1})$. As usual, the required fully $\mathcal{N}_{G_p}$-triangular subdiagrams with isolated edges

labeled by $t^{\pm 1}$ have been built in case 1, and the degenerate diagram $\Delta_{u,a^{-1}}$ is given in case 2.1. Since $\mathsf{nf}(ua^{-1}) = u'$ is a prefix of $u$, we again have built the diagram $\Delta_{\mathsf{nf}(ua^{-1}),s^{-1}}$ by induction. So it suffices to show that fully $\mathcal{N}_{G_p}$-triangular van Kampen diagrams have been built for the pairs $(u_1, a)$, and $(u_2, a)$, where $u_1 = \mathsf{nf}(ua^{-1}s^{-1})$ and $u_2 = \mathsf{nf}(ua^{-1}s^{-1}at)$.

Applying the property noted at the start of case 4, the normal form $u_1$ satisfies $\mathsf{tl}(u_1) = \mathsf{nf}(\mathsf{tl}(u)s^{-1})$ and

$$\mathsf{hd}(u_1) = \mathsf{nf}([s(t^{r_u}a^{\alpha_{r_u,u}}t^{-r_u})\cdots(t^{l_u}a^{\alpha_{l_u,u}-1}t^{-l_u})s^{-1}]t^{l_u}).$$

In the polynomial view, conjugation by $\hat{s}$ results in multiplication by $1 + x$, and so $q_{u_1}(x) = [q_u(x) - x^{l_u}](1 + x)$. Now $m_{u_1} = l_u$ and $l_{u_1} \le l_u + 1$, and so $m_{u_1} - l_{u_1} \ge -1$. Hence the diagram $\Delta_{u_1,a}$ is constructed case 2 or 3.

A similar computation yields $q_{u_2}(x) = [q_u(x) - x^{l_u}](1 + x) + x^{l_u}$, which also has degree $l_{u_2} \le l_u + 1$, and $m_{u_2} = l_u + 1$, and so $m_{u_2} - l_{u_2} \ge 0$. Hence the diagram $\Delta_{u_2,a}$ is constructed in case 2.

**Case 5**: *Suppose that $z = s$ and $\mathsf{hd}(u) \in \operatorname{Im}\psi$.* Similar to the situation in case 4, a property that we exploit in this case is that $\mathsf{tl}(\mathsf{nf}(\mathsf{tl}(u)sh)) = \mathsf{nf}(\mathsf{tl}(u)s)$ and $\mathsf{hd}(\mathsf{nf}(\mathsf{tl}(u)sh)) = \mathsf{nf}(h)$ for all $h \in \{a^{\pm 1}, t^{\pm 1}\}^*$.

Observe that $\mathsf{hd}(u) \in \operatorname{Im}\psi$ implies that $q_u(x) = \hat{q}_u(x)(1 + x)$ for some $\hat{q}_u(x) \in \mathbb{Z}_p[x, \frac{1}{x}]$ such that either $q_u = \hat{q}_u = 0$, or else $|l_u - r_u| > 0$ and

$$\hat{q}_u(x) = \gamma_{r_u,u}x^{r_u} + \cdots + \gamma_{l_u-1,u}x^{l_u-1}$$

with each $\gamma_{i,u} \in \mathbb{Z}_p$, and $\gamma_{l_u-1,u} = \alpha_{l_u,u}$.

We proceed by induction on the number $\mathsf{occ}_t(u)$ of occurrences of $t^{\pm 1}$ in the word $\mathsf{hd}(u)$.

First suppose that $\mathsf{occ}_t(u) = 0$. Since $q_u(x)$ is not a single monomial, $\mathsf{hd}(u)$ cannot be a nontrivial power of $a$, and so $\mathsf{hd}(u) = 1$ and $u = \mathsf{tl}(u)$. The normal form $\mathsf{nf}(us)$ is either $us$ or the word $u$ with a final letter $s^{-1}$ removed. In this case there is a degenerate diagram $\Delta_{u,z}$, and we set $\Phi(e_{u,z}) := e_{u,z}$.

Now suppose that $\mathsf{occ}_t(u) > 0$, and write $u = u'z'$ with $u' \in \mathcal{N}_{G_p}$ and $z' \in \{a, t^{\pm 1}\}$.

If $z' = t^\delta$ with $\delta \in \{t^{\pm 1}\}$, then as in case 4 we begin the diagram $\Delta_{u,z}$ with an isolated cell labeled by $zt^{-\delta}s^{-1}t^\delta$ and set $\Phi(e_{u,z}) := \mathsf{path}(u, t^{-\delta}st^\delta)$. The required fully $\mathcal{N}_{G_p}$-triangular diagrams with isolated edges labeled $t^{\pm 1}$ are constructed in case 1, and $\Delta_{u',z}$ has already been built by induction.

Suppose instead that $z' = a$. We build $\Delta_{u,z}$ with an isolated cell labeled $zt^{-1}a^{-\alpha_{l_u,u}}s^{-1}a^{\alpha_{l_u,u}}ta^{\alpha_{l_u,u}}$; then $\Phi(e_{u,z}) := \mathsf{path}(u, a^{-\alpha_{l_u,u}}t^{-1}a^{-\alpha_{l_u,u}}sa^{\alpha_{l_u,u}}t)$. For all $0 \le i \le \alpha_{l_u,u} - 1$, the diagram is degenerate as in case 2.1 thus $\Phi(u, z) := e_{u,z}$. and by case 1 it remains to show that we have built fully $\mathcal{N}_{G_p}$-triangular diagrams associated to the pairs $(u_{1,i}, a^{-1})$, $(u_2, s)$, and $(u_{3,i}, a)$ for $0 \le i < \alpha_{l_u,u}$, where $u_{1,i} = \mathsf{nf}(ua^{-\alpha_{l_u,u}}t^{-1}a^{-i})$, $u_2 = \mathsf{nf}(ua^{-\alpha_{l_u,u}}t^{-1}a^{-\alpha_{l_u,u}})$, and $u_{3,i} = \mathsf{nf}(ua^{-\alpha_{l_u,u}}t^{-1}a^{-\alpha_{l_u,u}}sa^i)$.

Since $q_u \ne 0$, then $|l_u - r_u| > 0$ and we can write $u = \mathsf{tl}(u)u''a^j ta^{\alpha_{l_u,u}}$ for some $u'' \in \mathcal{N}_{H_p}$, $j \in \{0, 1, ..., p-1\}$ and $\alpha_{l_u,u} \in \{1, 2, ..., p-1\}$ such that $u''$ does not end with $a$. Then $u_{1,i} = \mathsf{tl}(u)u''a^{\tilde{j}}$ where $\tilde{j} \in \{0, ..., p-1\}$

and $\tilde{j} \equiv j - i \pmod{p}$. Thus either $q_{u_{1,i}}(x) = 0$, or else $q_{u_{1,i}} \neq 0$ and $m_{u_{1,i}} - l_{u_{1,i}} \geq 0$, and so $\Delta_{u_{1,i},a^{-1}}$ is constructed in case 2.

Next note that $q_{u_2} = q_u - \alpha_{l_u,u} x^{l_u-1}(x+1)$, and so $\mathsf{hd}(u_2) \in \mathrm{Im}\,\psi$ and the pair $(u_2, s)$ satisfies case 5. Moreover, $u_2 = \mathsf{tl}(u)u''a^{\tilde{j}}$ where $\tilde{j} \equiv j - \alpha_{l_u,u} \pmod{p}$, and so $\mathsf{occ}_t(u_2) = \mathsf{occ}_t(u) - 1 < \mathsf{occ}_t(u)$. Hence the diagram $\Delta_{u_2,s}$ has already been built by induction.

Applying the property noted at the beginning of case 5, we have that $\mathsf{tl}(u_{3,i}) = \mathsf{nf}(\mathsf{tl}(u)s)$ and

$$\mathsf{hd}(u_{3,i}) = \mathsf{nf}([s^{-1}(t^{r_u}a^{\alpha_{r_u,u}}t^{-r_u})\cdots(t^{l_u-1}a^{\alpha_{l_u-1,u}-\alpha_{l_u,u}}t^{-(l_u-1)})s](t^{l_u-1}a^i t^{-(l_u-1)})t^{l_u-1}).$$

In the polynomial viewpoint, conjugation by $\hat{s}^{-1}$ is division by $1 + x$, and so $q_{u_{3,i}} = \hat{q}_u - \alpha_{l_u,u}x^{l_u-1} + ix^{l_u-1}$. Now either $q_{u_{3,i}} = 0$ (in the case that $i = 0$ and $q_u = \alpha_{l_u,u}x^{l_u-1}(x+1))$ or else $q_{u_{3,i}} \neq 0$ and $l_{u_{3,i}} \leq l_u - 1 = m_{u_{3,i}}$ (and so $m_{u_{3,i}} - l_{u_{3,i}} \geq 0$). Thus the diagram $\Delta_{u_{3,i},a}$ is constructed in case 2.

**Case 6**: *Suppose that $z = a^{\pm 1}$, $q_u(x) \neq 0$, and $m_u - l_u < -1$. Write $z = a^\delta$.* The Laurent polynomial $q_u(x) = \sum_{i=r_u}^{l_u} \alpha_{i,u}x^i$ can be written

$$q_u(x) = \hat{q}_u(x)(1+x) + R_u$$

for some Laurent polynomial $\hat{q}_u(x) \in \mathbb{Z}_p[x, \frac{1}{x}]$ and $R_u = q_u(-1) \in \mathbb{Z}_p$. Define

$$d_\delta(u) = (d_{1,\delta}(u), d_{2,\delta}(u)) := (|m_u - l_u|, R_u - \delta(-1)^{m_u+1}) \in \{i \in \mathbb{N} \mid i \geq 2\} \times \{0, ..., p-1\},$$

where we consider the element $R_u - \delta(-1)^{m_u+1}$ of $\mathbb{Z}_p$ to be an integer in $\{0, ..., p-1\} \subset \mathbb{N}_0$. Consider the lexicographic ordering $\prec$ on $\{i \in \mathbb{N} \mid i \geq 2\} \times \{0, ..., p-1\} \subset \mathbb{N} \times \mathbb{N}_0$ which is a well-founded strict partial ordering. We proceed by (Noetherian) induction on this ordering on $d_\delta(u)$.

For the base case, suppose that $d_\delta(u) = (2, 0)$. Then $m_u - l_u = -2$ and $q_u(-1) - \delta(-1)^{m_u+1} = 0$. We begin the diagram $\Delta_{u,z}$ with an isolated cell with an isolated edge labeled $z$, and edge path in the other direction labeled $ta^{-\delta}t^{-1}sa^\delta s^{-1}$; that is, $\Phi(e_{u,z}) := \mathsf{path}(u, ta^{-\delta}t^{-1}sa^\delta s^{-1})$. The subdiagrams $\Delta_{u,t}$ and $\Delta_{\mathsf{nf}(uta^{-\delta}),t^{-1}}$ are constructed in case 1; there are four remaining subdiagrams needed to build $\Delta_{u,z}$ that we need to show have already been constructed: $\Delta_{u_1,a^{-\delta}}$, $\Delta_{u_2,s}$, $\Delta_{u_3,a^\delta}$, and $\Delta_{u_4,s^{-1}}$, where $u_1 = \mathsf{nf}(ut)$, $u_2 = \mathsf{nf}(uta^{-\delta}t^{-1})$, $u_3 = \mathsf{nf}(uta^{-\delta}t^{-1}s)$, and $u_4 = \mathsf{nf}(uta^{-\delta}t^{-1}sa^\delta)$.

Since $m_{u_1} - l_{u_1} = -1$, the diagram $\Delta_{u_1,a^{-\delta}}$ is built in case 3.

The polynomial associated to $u_2$ is $q_{u_2}(x) = q_u(x) - \delta x^{m_u+1}$, which satisfies $q_{u_2}(-1) = d_{2,\delta}(u) = 0$. Then $1 + x$ divides the Laurent polynomial $q_{u_2}$ and $\mathsf{hd}(u_2) \in \mathrm{Im}\,\psi$. Hence the diagram $\Delta_{u_2,s}$ is built in case 5.

Next, as in case 5 we note that $\mathsf{tl}(u_3) = \mathsf{nf}(\mathsf{tl}(u)s)$ and

$$\mathsf{hd}(u_3) = \mathsf{nf}([s^{-1}(t^{r_u}a^{\alpha_{r_u,u}}t^{-r_u})\cdots(t^{l_u}a^{\alpha_{l_u,u}}t^{-l_u})(t^{m_u+1}a^{-\delta}t^{-(m_u+1)})s]t^{m_u}).$$

Then $q_{u_3}(x) = (q_u(x) - \delta x^{m_u+1})/(x+1)$; since $m_u + 1 < l_u$, the degree of $q_{u_3}$ is $l_{u_3} = l_u - 1$. Since $m_{u_3} = m_u$, then $m_{u_3} - l_{u_3} = m_u - l_u + 1 = -1$, and $\Delta_{u_3,a^\delta}$ is also constructed in case 3.

Finally, we also have $\mathsf{tl}(u_4) = \mathsf{nf}(\mathsf{tl}(u)s)$, and so $\mathsf{tl}(u_4)$ cannot end with $as$. Hence $\Delta_{u_4,s^{-1}}$ is constructed in case 4.

For the inductive step, suppose that $d_\delta(u) \succ (2,0)$.

Suppose further that $d_{2,\delta}(u) = 0$ (and hence $d_{1,\delta}(u) = |m_u - l_u| > 2$). We follow nearly the same proof as in the $d_\delta(u) = (2,0)$ (base) case above; the isolated cell of $\Delta_{u,z}$ is labeled $zsa^{-\delta}s^{-1}ta^\delta t^{-1}$, and $\Phi(e_{u,z}) := \mathsf{path}(u, ta^{-\delta}t^{-1}sa^\delta s^{-1})$. The only differences with the proof of that base case is that the applications of case 3 are replaced with induction. In particular, $u_1$ satisfies $m_{u_1} - l_{u_1} = m_u - l_u + 1$, implying that $d_{1,\delta}(u_1) = |m_{u_1} - l_{u_1}| = |m_u - l_u| + 1 < |m_u - l_u| = d_{1,\delta}(u)$, and so $d_\delta(u_1) \prec d_\delta(u)$ and the construction of $\Delta_{u_1,a^{-\delta}}$ follows from induction. Similarly the fact that $m_{u_3} - l_{u_3} = m_u - l_u + 1$ implies that $\Delta_{u_3,a^\delta}$ is built by induction.

On the other hand suppose that $d_{2,\delta}(u) > 0$ and let $\eta := (-1)^{m_u+1}$. The construction of $\Delta_{u,z}$ begins with an isolated cell having isolated edge labeled $z$, and edge path in the other direction labeled $ta^{-\eta}t^{-1}a^\delta ta^\eta t^{-1}$ then $\Phi(e_{u,z}) := \mathsf{path}(u, ta^{-\eta}t^{-1}a^\delta ta^\eta t^{-1})$. Applying case 1 to obtain the required (degenerate) diagrams with isolated edges labeled $t^{\pm 1}$, we have left to check whether the following three fully $\mathcal{N}_{G_p}$-triangular diagrams $\Delta_{u_1,a^{-\eta}}$, $\Delta_{\tilde{u}_2,a^\delta}$ and, $\Delta_{\tilde{u}_3,a^\eta}$ have already been constructed, where $u_1 = \mathsf{nf}(ut)$, $\tilde{u}_2 = \mathsf{nf}(uta^{-\eta}t^{-1})$, and $\tilde{u}_3 = \mathsf{nf}(uta^{-\eta}t^{-1}a^\delta t)$.

Since $d_{1,\delta}(u_1) = d_{1,\delta}(u)+1$, then $\Delta_{u_1,a^{-\eta}}$ is constructed by case 3 or induction. Next $q_{\tilde{u}_2}(x) = q_u(x) - \eta x^{m_u+1}$. Again using the fact that $m_u + 1 < l_u$, then $q_{\tilde{u}_2}(x) \neq 0$, $m_{\tilde{u}_2} = m_u$, and $l_{\tilde{u}_2} = l_u$; hence $d_{1,\delta}(\tilde{u}_2) = d_{1,\delta}(u)$. Moreover,

$$d_{2,\delta}(\tilde{u}_2) = R_{\tilde{u}_2} - \delta(-1)^{m_{\tilde{u}_2}} = R_u - \eta(-1)^{m_u+1} - \delta(-1)^{m_u} = d_{2,\delta}(u) - 1.$$

Therefore the construction of $\Delta_{\tilde{u}_2,a^\delta}$ follows from induction. Finally $q_{\tilde{u}_3}(x) = q_u(x) - \eta x^{m_u+1} + \delta x^{m_u}$, $m_{\tilde{u}_3} = m_u + 1$, and $l_{\tilde{u}_3} = l_u$. Then $d_{1,\delta}(\tilde{u}_3) < d_{1,\delta}(u)$, and again the construction of $\Delta_{\tilde{u}_3,a^\eta}$ follows from induction.

**Case 7**: *Suppose that $z = s^{-1}$ and $\mathsf{tl}(u)$ has a suffix $as$.* We proceed by induction on the length $l(\mathsf{hd}(u))$.

Suppose first that $l(\mathsf{hd}(u)) = 0$. Then there is a degenerate diagram $\Delta_{u,z}$ and $\Phi(e_{u,z}) := e_{u,z}$.

Suppose next that $l(\mathsf{hd}(u)) > 0$. Write $u = u'z'$ with $u' \in \mathcal{N}_{G_p}$ and $z' \in \{a, t^{\pm 1}\}$.

If $z' = t^\delta$ with $\delta \in \{t^{\pm 1}\}$, the construction and inductive proof are identical to the $z' = t^\delta$ subcase of case 4, with $\Phi(e_{u,z}) := \mathsf{path}(u, t^{-\delta}st^\delta)$.

On the other hand suppose that $z' = a$. We begin building $\Delta_{u,z}$ with an isolated cell labeled $zta^{-1}t^{-1}a^{-1}sa$; that is, $\Phi(eu, z) := \mathsf{path}(u, a^{-1}s^{-1}atat^{-1})$. Now case 1 provides fully $\mathcal{N}_{G_p}$-triangular van Kampen diagrams $\Delta_{\hat{u},t^\delta}$ and cases 2,3, and 6 provide fully $\mathcal{N}_{G_p}$-triangular van Kampen diagrams $\Delta_{\hat{u},a^\delta}$ for all $\hat{u} \in \mathcal{N}_{G_p}$ and $\delta \in \{\pm 1\}$, yielding diagrams $\Delta_{u,a^{-1}}$, $\Delta_{\mathsf{nf}(ua^{-1}s^{-1}),a}$, $\Delta_{\mathsf{nf}(ua^{-1}s^{-1}a),t}$, $\Delta_{\mathsf{nf}(ua^{-1}s^{-1}at),a}$, and $\Delta_{\mathsf{nf}(ua^{-1}s^{-1}ata),t^{-1}}$. Since $\mathsf{nf}(ua^{-1}) = u'$ satisfies $l(\mathsf{hd}(u')) = l(\mathsf{hd}(u)) - 1 < l(\mathsf{hd}(u))$, the diagram $\Delta_{\mathsf{nf}(ua^{-1}),s^{-1}}$ has also already been constructed, by induction.

**Case 8**: *Suppose that $z = s$ and $\mathsf{hd}(u) \notin \operatorname{Im} \psi$*. As in case 5, we proceed by induction on the number $\mathsf{occ}_t(u)$ of occurrences of $t^{\pm 1}$ in $\mathsf{hd}(u)$.

Suppose first that $\mathsf{occ}_t(u) = 0$. Then $\mathsf{hd}(u) = a^{\alpha_{l_u}}$, and since $\mathsf{hd}(u) \notin \operatorname{Im} \psi$, then $\alpha_{l_u} > 0$. In this case the word $us$ is in normal form, and so we have $\Phi(e_{u,z}) := e_{u,z}$.

Now suppose that $\mathsf{occ}_t(u) > 0$, and write $u = u'z'$ with $u' \in \mathcal{N}_{G_p}$ and $z' \in \{a, t^{\pm 1}\}$.

If $z' = t^\delta$ with $\delta \in \{t^{\pm 1}\}$, the construction of $\Delta_{u,z}$ has isolated cell labeled $zt^{-\delta}s^{-1}t^\delta$, with $\Phi(e_{u,z}) := \mathsf{path}(u, t^{-\delta}st^\delta)$. Since $\mathsf{nf}(ut^{-\delta}) = u'$ satisfies $\mathsf{occ}_t(u') = \mathsf{occ}_t(u) - 1 < \mathsf{occ}_t(u)$, the diagram $\Delta_{u',z}$ has already been constructed by induction.

Suppose instead that $z' = a$. Now $\mathsf{hd}(u) = u''t^\delta a^{\alpha_{l_u}}$ for some $\delta \in \{\pm 1\}$ and $u'' \in \mathcal{N}_{H_p}$. Again we recall that $p < \infty$ in Step II, and so $\alpha_{l_u} \in \{1, 2, ..., p - 1\}$.

If $\delta = 1$, we build $\Delta_{u,z}$ with an isolated cell labeled $zt^{-1}a^{-\alpha_{l_u}}s^{-1}a^{\alpha_{l_u}}ta^{\alpha_{l_u}}$; then $\Phi(e_{u,z}) := \mathsf{path}(u, a^{-\alpha_{l_u}}t^{-1}a^{-\alpha_{l_u}}sa^{\alpha_{l_u}}t)$. Cases 1,2,3, and 6 provide fully $\mathcal{N}_{G_p}$-triangular van Kampen diagrams corresponding to the edges labeled $t^{\pm 1}$ and $a^{\pm 1}$ along the boundary of this isolated cell. Now the normal form $\tilde{u} := \mathsf{nf}(ua^{-\alpha_{l_u}}t^{-1}a^{-\alpha_{l_u}}) = \mathsf{nf}(u''a^{-\alpha_{l_u}})$ is obtained from $u''a^{-\alpha_{l_u}}$ by possible reduction modulo $p$ of a final power of $a$; then $\tilde{u}$ satisfies $\mathsf{occ}_t(\tilde{u}) = \mathsf{occ}_t(u) - 1 < \mathsf{occ}_t(u)$, and again induction applies to show that the diagram $\Delta_{\tilde{u},s}$ has already been constructed.

If $\delta = -1$, we build $\Delta_{u,z}$ with an isolated cell labeled $za^{-\alpha_{l_u}}t^{-1}s^{-1}a^{\alpha_{l_u}}t^{-1}a^{\alpha_{l_u}}$; then $\Phi(e_{u,z}) = \mathsf{path}(u, a^{-\alpha_{l_u}}ta^{-\alpha_{l_u}}st^{-1}a^{\alpha_{l_u}})$. As before, cases 1,2,3, and 6 provide fully $\mathcal{N}_{G_p}$-triangular van Kampen diagrams corresponding to the edges labeled $t^{\pm 1}$ and $a^{\pm 1}$. The normal form $\hat{u} := \mathsf{nf}(ua^{-\alpha_{l_u}}ta^{-\alpha_{l_u}}) = u''a^{-\alpha_{l_u}}$, and so $\mathsf{occ}_t(\tilde{u}) < \mathsf{occ}_t(u)$. Once again induction applies to show that the diagram $\Delta_{\hat{u},s}$ has already been constructed, as required.

*Step III. Autostackability of $G_p$ for $p < \infty$:*

In this section again we consider the $p < \infty$ case. Throughout this step we will repeatedly apply the closure properties of regular and synchronously regular languages discussed in Section 2.

Before analyzing the graph of the stacking map, we first discuss the set $\mathcal{N}_{G_p}$ of normal forms. Let

$$\mathsf{Tail} \quad := \quad \{s^{-k}a^{\beta_1}sa^{\beta_2}s \ldots sa^{\beta_n}s \mid k, n \geq 0, \beta_i \in \mathbb{Z}_p \text{ for } 1 \leq i \leq n,$$
$$\text{and } \beta_1 \neq 0 \text{ if both } k > 0 \text{ and } n > 0\};$$

that is, $\mathsf{Tail}$ is the set of tails of elements of $\mathcal{N}_{G_p}$. Then

$$\mathsf{Tail} = (s^{-1})^*(\{1, a, a^2, ..., a^{p-1}\}s)^* \smallsetminus X^*s^{-1}sX^*$$

is built from finite subsets of $X^*$ using concatenation, complement, and Kleene star operations, and hence is a regular language. Similarly the normal form set $\mathcal{N}_{H_p}$ for $H_p$ can be written

$$\mathcal{N}_{H_p} = (t^* \cup (t^{-1})^*)(\{1, a, a^2, ..., a^{p-1}\}t)^*\{1, a, a^2, ..., a^{p-1}\}(t^* \cup (t^{-1})^*) \smallsetminus X^*\{tt^{-1}, t^{-1}t\}X^*$$

and so $\mathcal{N}_{H_p}$ is also regular. Finally the normal form set $\mathcal{N}_{G_p}$ is the concatenation $\mathcal{N}_{G_p} = \mathsf{Tail} \cdot \mathcal{N}_{H_p}$ of these two regular languages, and therefore $\mathcal{N}_{G_p}$ is also regular.

The stacking map associated to the flow function $\Phi$ in Step II of this proof is given by

$$\phi(u,z) := \begin{cases} z & \text{if } z = t^{\pm 1} \end{cases} \tag{1}$$

from case 1,

$$\phi(u,z) := \begin{cases} z & \text{if } z = a \text{ and } u \in \mathsf{Tail} \cdot (t^{-1})^* \cup X^*t \cup (X^*a \smallsetminus X^*a^{p-1}) & (2) \\ a^{-(p-1)} & \text{if } z = a \text{ and } u \in X^*a^{p-1} & (3) \\ z & \text{if } z = a^{-1} \text{ and } u \in X^*a & (4) \\ a^{p-1} & \text{if } z = a^{-1} \text{ and } u \in \mathsf{Tail} \cdot (t^{-1})^* \cup X^*t & (5) \\ ta^{-1}t^{-1}a^\delta tat^{-1} & \text{if } z = a^\delta,\ \delta \in \{\pm 1\}, \text{ and } u \in X^*at^{-1} & (6) \\ ta^{-\delta}t^{-1}sa^\delta s^{-1} & \text{if } z = a^\delta,\ \delta \in \{\pm 1\}, \text{ and } u \in X^*aY^*t^{-2} \\ & \quad \text{and } q_u(-1) - \delta(-1)^{m_u+1} \equiv 0 (\text{mod } p) & (7) \\ ta^{-\eta}t^{-1}a^\delta ta^\eta t^{-1} & \text{if } z = a^\delta,\ \delta \in \{\pm 1\}, \text{ and } u \in X^*aY^*t^{-2}, \\ & \quad q_u(-1) - \delta(-1)^{m_u+1} \not\equiv 0 (\text{mod } p), \\ & \quad \text{and } \eta = (-1)^{m_u+1} & (8) \end{cases}$$

from cases 2,3, and 6 (where we recall that $Y = \{a^{\pm 1}, t^{\pm 1}\}$),

$$\phi(u,z) := \begin{cases} z & \text{if } z = s^{-1} \text{ and } u \in \mathsf{Tail} & (9) \\ t^{-\delta}s^{-1}t^\delta & \text{if } z = s^{-1} \text{ and } u \in X^*t^\delta,\ \delta \in \{\pm 1\} & (10) \\ a^{-1}s^{-1}atat^{-1} & \text{if } z = s^{-1} \text{ and } u \in X^*a & (11) \end{cases}$$

from cases 4 and 7, and

$$\phi(u,z) := \begin{cases} z & \text{if } z = s \text{ and } u \in \mathsf{Tail} \cdot a^* & (12) \\ t^{-\delta}st^\delta & \text{if } z = s \text{ and } u \in X^*t^\delta,\ \delta \in \{\pm 1\} & (13) \\ a^{-\alpha}t^{-1}a^{-\alpha}sa^\alpha t & \text{if } z = s \text{ and } u \in X^*ta^\alpha,\ \alpha \in \{1,...,p-1\} & (14) \\ a^{-\alpha}ta^{-\alpha}st^{-1}a^\alpha & \text{if } z = s \text{ and } u \in X^*t^{-1}a^\alpha,\ \alpha \in \{1,...,p-1\} & (15) \end{cases}$$

from cases 5 and 8.

The graph $\mathsf{graph}(\phi)$ of this function is a union of 15 languages $L_1, ..., L_{15}$, one for each of the pieces of this piecewise defined function. Each language $L_i$ is either a Cartesian product $L_i' \times \{z_i\} \times \{w_i\}$ for some $L_i' \subset X^*$, $z_i \in X$, and $w_i \in X^*$, or else is a finite union of such products; for example, $L_1 = \cup_{\delta \in \{\pm 1\}} L_{1\delta}' \times \{t^\delta\} \times \{t^\delta\}$, and $L_{1\delta}' = \mathcal{N}_{G_p}$. Since a finite union of a Cartesian product of regular languages is synchronously regular, it suffices to show that the sets $L_i'$ are regular.

For thirteen of the languages, regularity of the set $L_i'$ follows immediately from that of $\mathcal{N}_{G_p}$ and $\mathsf{Tail}$, along with standard closure properties of regular languages. In particular, $L_{1\delta}' = \mathcal{N}_{G_p}$, $L_2' = \mathcal{N}_{G_p} \cap (\mathsf{Tail} \cdot (t^{-1})^* \cup X^*t \cup (X^*a \smallsetminus X^*a^{p-1}))$, $L_3' = \mathcal{N}_{G_p} \cap X^*a^{p-1}$, $L_4' = \mathcal{N}_{G_p} \cap X^*a$, $L_5' = \mathcal{N}_{G_p} \cap \mathsf{Tail} \cdot (t^{-1})^* \cup X^*t$, $L_{6\delta}' = \mathcal{N}_{G_p} \cap X^*at^{-1}$, $L_9' = \mathsf{Tail}$, $L_{10\delta}' = \mathcal{N}_{G_p} \cap X^*t^\delta$, $L_{11}' = \mathcal{N}_{G_p} \cap X^*a$,

$L'_{12} = \mathsf{Tail} \cdot a^*$, $L'_{13\delta} = \mathcal{N}_{G_p} \cap X^* t^\delta$, $L'_{14\alpha} = \mathcal{N}_{G_p} \cap X^* t a^\alpha$, and $L'_{15\alpha} = \mathcal{N}_{G_p} \cap X^* t^{-1} a^\alpha$, for $\delta \in \{\pm 1\}$ and $\alpha \in \{1, ..., p-1\}$.

The remaining two subsets $L_7, L_8$ of $\mathsf{graph}(\phi)$ arise from case 6. The set $L_7$ is the union of languages $L'_{7\delta} \times \{a^\delta\} \times \{t a^{-\delta} t^{-1} s a^\delta s^{-1}\}$, and $L_8$ is the union of $L'_{8\delta\eta} \times \{a^\delta\} \times \{t a^{-\eta} t^{-1} a^\delta t a^\eta t^{-1}\}$, for $\delta, \eta \in \{\pm 1\}$, where

$$
\begin{aligned}
L'_{7\delta} &= \cup_{\eta \in \{\pm 1\}} X^* a Y^* t^{-2} \cap (\mathsf{Tail} \cdot (M_\eta \cap N_{\delta,\eta})) \text{ and} \\
L'_{8\delta\eta} &= X^* a Y^* t^{-2} \cap (\mathsf{Tail} \cdot (M_\eta \cap (\mathcal{N}_{H_p} \setminus N_{\delta,\eta}))), \text{ such that} \\
M_\eta &= \{u \in \mathcal{N}_{H_p} \mid (-1)^{m_u + 1} = \eta\} \text{ and} \\
N_{\delta,\eta} &= \{u \in \mathcal{N}_{H_p} \cap Y^* a Y^* t^{-1} \mid q_u(-1) \equiv \delta\eta \pmod{p}\}.
\end{aligned}
$$

In order to show that $L_7$ and $L_8$ are synchronously regular, it suffices to show that the languages $M_\eta$ and $N_{\delta,\eta}$ are regular.

For $u \in \mathcal{N}_{H_p}$, the integer $m_u$ is the sum of the exponents of the letters $t^{\pm 1}$ in $u$. That is, $M_1$ is the set of words in $\mathcal{N}_{H_p}$ with $t$-exponent sum odd, and $M_{-1}$ is the set of words in $\mathcal{N}_{H_p}$ with $t$-exponent sum even. Let $\gamma : Y^* \to \mathbb{Z}/2$ be the monoid homomorphism to the finite monoid $Z/2$ defined by $\gamma(a) = \gamma(a^{-1}) = 0$ and $\gamma(t) = \gamma(t^{-1}) = 1$. The preimage sets $\gamma^{-1}(\{0\})$ and $\gamma^{-1}(\{1\})$ are regular sets (see Section 2). Then $M_1 = \mathcal{N}_{H_p} \cap \gamma^{-1}(\{1\})$ and $M_{-1} = \mathcal{N}_{H_p} \cap \gamma^{-1}(\{0\})$ are intersections of regular languages, and hence $M_\eta$ is regular for $\eta \in \{\pm 1\}$.

Given any $u \in \mathcal{N}_{H_p} \cap Y^* a Y^* t^{-1}$, as usual we write $q_u(x) = \alpha_{r_u,u} x^{r_u} + \cdots + \alpha_{l_u,u} x^{l_u}$, where $u = t^{r_u} a^{\alpha_{r_u,u}} \cdots t a^{\alpha_{l_u,u}} t^{-l_u + m_u}$. Then $q_u(-1) = (-1)^{r_u}(\alpha_{r_u,u} + \cdots + \alpha_{l_u,u}(-1)^{l_u - r_u})$. Splitting this into separate cases depending upon whether $r_u$ is even or odd yields

$$
N_{\delta,\eta} = [((t^2)^* \cup (t^{-2})^*)\widetilde{N}_{\delta,\eta}(t^{-1})^*] \cup [(t(t^2)^* \cup t^{-1}(t^{-2})^*)\widetilde{N}_{-\delta,\eta}(t^{-1})^*]
$$

where

$$
\begin{aligned}
\widetilde{N}_{\delta,\eta} := \{&a^{\alpha_0} t a^{\alpha_1} \cdots t a^{\alpha_l} t^{-1} \mid 0 \le l, \ \alpha_i \in \{0, 1, ..., p-1\} \text{ for } 0 \le i \le l, \\
&\alpha_0, \alpha_l \ne 0, \text{ and } \alpha_0 - \alpha_1 + \cdots + (-1)^{\alpha_l} \alpha_l \equiv \delta\eta \pmod{p}\}.
\end{aligned}
$$

We note that addition and subtraction in the finite group $\mathbb{Z}/p\mathbb{Z}$ can be performed by a finite state automaton (see [10] and [15] for details on finite state automata) by encoding the group in the states of the automaton. It follows that the language $\widetilde{N}_{\delta,\eta}$ is regular. Using the closure properties for regular languages, then $N_{\delta,\eta}$ is also regular.

We now have that all of the languages $L_1, ..., L_{15}$ are synchronously regular. Therefore their union $\mathsf{graph}(\phi) = \cup_{i=1}^{15} L_i$ is also synchronously regular, as required.

*Step IV. Algorithmically stackable system of fully $\mathcal{N}_{G_p}$-triangular van Kampen diagrams for $p = \infty$:*

In this part of the proof we consider the case that $p = \infty$. The construction of the flow function can be done in a very similar way to the proofs in

earlier steps so we leave the details to the reader and just state the stacking map that one gets over the finite presentation

$$G_\infty = \langle a, s, t \mid a^p = 1, [a^t, a] = 1, [s, t] = 1, sas^{-1} = tat^{-1}a, sats^{-1} = ata \rangle.$$

$$\phi(u, z) := \begin{cases} z & \text{if } z = t^{\pm 1} \\ z & \text{if } z = a^{\pm 1} \text{ and } u \in \mathsf{Tail} \cdot (t^{-1})^* \cup X^* t \cup X^* a^{\pm 1} \\ ta^{-\varepsilon}t^{-1}a^\delta ta^\varepsilon t^{-1} & \text{if } z = a^\delta, \delta, \varepsilon \in \{\pm 1\}, \text{ and } u \in X^* a^\varepsilon t^{-1} \\ ta^{-\delta}t^{-1}sa^\delta s^{-1} & \text{if } z = a^\delta, \ \delta \in \{\pm 1\}, \text{ and } u \in X^* a^{\pm 1} Y^* t^{-2} \\ & \qquad \text{and } q_u(-1) - \delta(-1)^{m_u+1} = 0 \\ ta^{-\eta}t^{-1}a^\delta ta^\eta t^{-1} & \text{if } z = a^\delta, \ \delta \in \{\pm 1\}, \text{ and } u \in X^* a^{\pm 1} Y^* t^{-2}, \\ & \qquad q_u(-1) - \delta(-1)^{m_u+1} \neq 0 \\ & \qquad \text{and } \eta = (-1)^{m_u+1} \text{ if } q_u(-1) - \delta(-1)^{m_u+1} > 0, \\ & \qquad \eta = (-1)^{m_u} \text{ otherwise,} \\ z & \text{if } z = s^{-1} \text{ and } u \in \mathsf{Tail} \\ t^{-\delta}s^{-1}t^\delta & \text{if } z = s^{-1} \text{ and } u \in X^* t^\delta, \ \delta \in \{\pm 1\} \\ a^{-\delta}s^{-1}a^\delta ta^\delta t^{-1} & \text{if } z = s^{-1} \text{ and } u \in X^* a^\delta, \delta \in \{\pm 1\} \\ z & \text{if } z = s \text{ and } u \in \mathsf{Tail} \cdot (a^* \cup (a^{-1})^*) \\ t^{-\delta}st^\delta & \text{if } z = s \text{ and } u \in X^* t^\delta, \ \delta \in \{\pm 1\} \\ a^{-\delta}t^{-1}a^{-\delta}sa^\delta t & \text{if } z = s \text{ and } u \in X^* ta^\delta(a^\delta)^*, \ \delta \in \{\pm 1\} \\ a^{-\delta}ta^{-\delta}st^{-1}a^\delta & \text{if } z = s \text{ and } u \in X^* t^{-1}a^\delta(a^\delta)^*, \ \delta \in \{\pm 1\} \end{cases}$$

$\square$

Combining the results of Bartholdi, Neuhauser, and Woess [1], that the Diestel-Leader groups in Theorem 6.1 are not of homological type $\mathrm{FP}_3$, and of Groves and Smith [14], that the constructible metabelian groups are the metabelian groups of type $\mathrm{FP}_\infty$, shows that these Diestel-Leader groups are not constructible. Thus the following is immediate from Theorem 6.1.

**Corollary 6.2.** *The class of autostackable groups contains nonconstructible metabelian groups.*

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