A UNIFORM MODEL FOR ALMOST CONVEXITY AND REWRITING SYSTEMS

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ABSTRACT. We introduce a topological property for finitely generated groups called stackable that implies the existence of an inductive procedure for constructing van Kampen diagrams with respect to a particular finite presentation. We also define algorithmically stackable groups, for which this procedure is an algorithm. This property gives a common model for algorithms arising from both rewriting systems and almost convexity for groups.

1. INTRODUCTION

In geometric group theory, several properties of finitely generated groups have been defined using a language of normal forms together with geometric or combinatorial conditions on the associated Cayley graph, most notably in the concepts of combable groups and automatic groups in which the normal forms satisfy a fellow traveler property. In this paper we use a set of normal forms together with another topological property on the Cayley graph of a finitely generated group, namely a notion of "flow" toward the identity vertex, to define a property which we call stackable.

More specifically, let G be a group with a finite inverse-closed generating set A, and let $\Gamma = \Gamma(G, A)$ be the associated Cayley graph, with set $\vec{E}$ of directed edges. For each $g \in G$ and $a \in A$, let $e_{g,a}$ denote the directed edge in $\vec{E}$ with initial vertex g, terminal vertex ga, and label a. Given a set $\mathcal{N} \subset A^*$ of normal forms for G over A, write y_g for the normal form of the element g of G. Note that in the context of this paper each element has a unique normal form; the map $y_g \mapsto g$ is a bijection. Note also that whenever an equality of words $y_g a = y_{ga}$ or $y_g = y_{ga} a^{-1}$ holds, there is a van Kampen diagram for the word $y_g a y_{ga}^{-1}$ that contains no 2-cells; in this case, we call the edge $e_{g,a}$ *degenerate*. Let $\vec{E}_d = \vec{E}_{d,\mathcal{N}}$ be the set of degenerate edges, and let $\vec{E}_r = \vec{E}_{r,\mathcal{N}} := \vec{E} \setminus \vec{E}_d$.

Definition 1.1. *Let G be a finitely generated group. The group G is* stackable *with respect to a finite inverse-closed generating set A if there exist a set $\mathcal{N}$ of normal forms for G over A containing the empty word, a well-founded strict partial ordering $<$ on $\vec{E}_r$, and a constant k, such that for every $g \in G$ and $a \in A$, there exists a path p from g to ga in Γ of length at most k satisfying the property that whenever e' is a directed edge in the path p, either $e', e_{g,a} \in \vec{E}_r$ and $e' < e_{g,a}$, or else $e' \in \vec{E}_d$.*

In Section 3 we show that a stackable group is finitely presented, and the set $\mathcal{N}$ of normal forms for a stackable group is prefix-closed.

Proposition 3.1 *Let G be a stackable group. Then G is finitely presented, and the set of normal forms of the stackable structure is closed under taking prefixes.*

Thus $\mathcal{N}$ determines a maximal tree in Γ, namely the edges lying on paths that are labeled by words in $\mathcal{N}$ and that start at the vertex labeled by the identity of G. This leads to the following topological description of stackability. Let $\vec{P}$ denote the set of directed paths in Γ. For each $g \in G$ and $a \in A$, we view the two directed edges $e_{g,a}$ and $e_{ga,a^{-1}}$ of Γ to have a single underlying undirected edge in Γ.

Definition 1.2. *A* flow function *associated to a maximal tree $\mathcal{T}$ in the Cayley graph Γ is a function $\Phi : \vec{E} \to \vec{P}$ satisfying the properties that:*

(F1) *For each edge $e \in \vec{E}$, the path $\Phi(e)$ has the same initial and terminal vertices as e.*
(F2d) *If the undirected edge underlying e lies in the tree $\mathcal{T}$, then $\Phi(e) = e$.*
(F2r) *The transitive closure $<_{\Phi}$ of the relation $<$ on $\vec{E}$, defined by*

        $e' < e$ whenever e' lies on the path $\Phi(e)$ and the undirected edges underlying both e and e' do not lie in $\mathcal{T}$,

    is a well-founded strict partial ordering.

Moreover, the flow function Φ is bounded *if there is a constant k such that for all $e \in \vec{E}$, the path $\Phi(e)$ has length at most k.*

Informally, the map Φ fixes the edges lying in the tree $\mathcal{T}$ and describes a "flow" of the non-tree edges toward the tree (or toward the basepoint); starting from a non-tree edge and iterating this function finitely many times results in a path in the tree.

Corollary 3.2. *A group G is stackable with respect to a finite symmetric generating set A if and only if the Cayley graph $\Gamma(G, A)$ admits a bounded flow function.*

The two equivalent descriptions of stackability in Corollary 3.2 are written to display connections to two other properties exploited later in the paper: Definition 1.1 closely resembles Definition 4.3 of almost convexity, and the bounded flow function is analogous to rewriting operations.

We show that every stackable group admits an inductive procedure which, upon input of a word in the generators that represents the identity of the group, constructs a van Kampen diagram for that word over the finite presentation from Proposition 3.1. These van Kampen diagrams for stackable groups are constructed by building up stacks of van Kampen diagrams associated to recursive edges, leading to the terminology "stackable groups". Let $\rho : A^* \to G$ be the canonical monoid homomorphism and let $\lambda : \vec{P} \to A^*$ map each path in Γ to the word labeling its edges. The following stronger property guarantees that the inductive procedure for constructing van Kampen diagrams is an algorithm.

Definition 1.3. *A group G with a finite symmetric generating set A is* algorithmically stackable *if there is a bounded flow function Φ on $\Gamma(G, A)$ for which the subset*

$$S_{\Phi} := \{(w, a, \lambda(\Phi(e_{\rho(w),a}))) \mid w \in A^*, a \in A\}$$

of $A^* \times A \times A^*$ is recursive.

Theorem 3.3. *If G is algorithmically stackable, then G has solvable word problem.*

The stackable property provides a uniform model for procedures for building van Kampen diagrams arising from two other group theoretic properties that yield algorithms to solve the word problem, namely finite complete rewriting systems and almost convexity. We discuss these and other examples of stackable structures for groups in Section 4. To begin, in Section 4.1 we give explicit details of a bounded flow function for the Baumslag-Solitar group $BS(1, p)$ with $p \geq 3$.

In Section 4.2 we consider groups that can be presented by rewriting systems. A *finite complete rewriting system* for a group G consists of a finite set A and a finite subset $R \subseteq A^* \times A^*$ such that as a monoid, G is presented by $G = Mon\langle A \mid u = v$ whenever $(u, v) \in R\rangle$, and the rewritings $xuy \to xvy$ for all $x, y \in A^*$ and $(u, v) \in R$ satisfy the properties that the words that cannot be rewritten are a set of normal forms for G, and every sequence of rewritings terminates after finitely many steps. Such a system gives a solution to the word problem for the group; given any word w over A, the word w represents the identity element of G if and only if any (and hence every) rewriting of w terminates at the empty word. (See [10] or [22] for more information about rewriting systems for groups and their applications.) Examples of groups with finite complete rewriting systems include finite groups, surface groups [19], constructible solvable groups [12], tame prime alternating [6] and torus [9] knot groups, and Artin groups of finite type [14]; this class is closed under group extensions [12],[14] and graph products [13].

Theorem 4.1. *A group admitting a finite complete rewriting system is algorithmically stackable.*

A flow function can be viewed as giving directions pointing from edges toward the basepoint vertex labeled by the identity ϵ of G, in the Cayley complex for the stacking presentation. (For example, an illustration of this flow for the Baumslag-Solitar group $BS(1, 2)$ is given in Figure 5 in Section 4.1.) That is, from any degenerate edge one can follow the maximal tree $\mathcal{T}$ associated to $\mathcal{N}$ to the next edge e' along the unique simple path toward ϵ, and from any recursive edge e, one can follow a 2-cell to a path containing edges e' each of which is either degenerate, or else is recursive and satisfies $e' <_\Phi e$. In both cases, we view e' as "closer" than e to the basepoint. A natural special case to consider occurs when this notion of "closer" coincides with the path metric d_X on the Cayley graph $X^1 = \Gamma$. That is, define the function $\alpha : \vec{E} \to \mathbb{Q}$ by setting $\alpha(e) := \frac{1}{2}(d_X(\epsilon, a) + d_X(\epsilon, b))$ for each edge $e \in \vec{E}(X)$ with endpoints a and b, so that α measures the average distance from a point of e to the origin.

Definition 1.4. *A group G is* geodesically stackable *if G has a finite symmetric generating set A with a stackable structure over a normal form set $\mathcal{N}$ and an associated bounded flow function Φ such that all of the elements of $\mathcal{N}$ label geodesic paths in $\Gamma(G, A)$, and whenever $e', e \in \vec{E}_r$ with $e' <_\Phi e$, then $\alpha(e') < \alpha(e)$.*

In Section 4.4 we show that this property is equivalent to Cannon's almost convexity property [5]. Informally, a finitely generated group G is almost convex if there is a constant bound on the extent to which balls centered at the identity in the Cayley graph fail to be convex; see Definition 4.3 for complete details. Similar to the situation for stackability, the

definition of almost convexity only requires a finite generating set, but Cannon has shown that almost convex groups are finitely presented, and has given an algorithm to solve the word problem for these groups [5]. The proof of the equivalence between almost convexity and geodesic stackability in Section 4.4 also yields the somewhat unexpected result that any geodesically stackable structure can be replaced by another that is both algorithmic and based upon the shortlex normal forms.

Theorem 4.4. *Let G be a group with finite symmetric generating set A. The following are equivalent:*

(1) *The pair (G, A) is almost convex.*
(2) *The pair (G, A) is geodesically stackable.*
(3) *The pair (G, A) is geodesically algorithmically stackable with respect to shortlex normal forms.*

The properties in Theorem 4.4 are satisfied by all word hyperbolic groups and cocompact discrete groups of isometries of Euclidean space, with respect to every generating set [5]. Hence Theorem 4.4 shows that every word hyperbolic group is algorithmically stackable.

One of the motivations for the definition of automatic groups was to gain a better understanding of the fundamental groups of 3-manifolds, in particular to find practical methods for computing in these groups. However, the fundamental group of a 3-manifold is automatic if and only if its JSJ decomposition does not contain manifolds with a uniform Nil or Sol geometry [10, Theorem 12.4.7]. In contrast, [17] Hermiller and Shapiro have shown that the fundamental group of every closed 3-manifold with a uniform geometry other than hyperbolic must have a finite complete rewriting system, and so combining this result with Theorems 4.1 and 4.4 yields the following.

Corollary 1.5. *If G is the fundamental group of a closed 3-manifold with a uniform geometry, then G is algorithmically stackable.*

The algorithmically stackable property also allows a wider range of Dehn (or isoperimetric) functions than those for automatic or combable groups, whose Dehn functions are at most quadratic [10] or exponential (shown by Gersten; see, for example, [2]), respectively. In particular, the iterated Baumslag-Solitar group

$$G_k = \langle a_0, a_1, ..., a_k \mid a_i^{a_{i+1}} = a_i^2; 0 \le i \le k-1 \rangle$$

admits a finite complete rewriting system for each $k \ge 1$ (first described by Gersten; see [15] for details), and so Theorem 4.1 shows that G_k is algorithmically stackable. Gersten [11, Section 6] showed that the Dehn function for G_k grows at least as fast as the function

$$n \mapsto \underbrace{2^{2^{\cdot^{\cdot^{2^n}}}}}_{k \text{ times}} .$$

Hence the class of algorithmically stackable groups includes groups whose Dehn functions are towers of exponentials.

In Section 4.3, we consider Thompson's group F; that is, the group of orientation-preserving piecewise linear automorphisms of the unit interval for which all linear slopes

are powers of 2, and all breakpoints lie in the the 2-adic numbers. In [7], Cleary, Hermiller, Stein, and Taback show that Thompson's group F is stackable (although they do not use this terminology, they build a stackable structure in their construction of a 1-combing for F), and their proof can be shown to give algorithmic stackability. We show in Section 4.3 that the set of normal forms associated to this stackable structre is a deterministic context-free language. Thompson's group F has been the focus of considerable research in recent years, and yet the questions of whether F has a finite complete rewriting system or is automatic are open (see the problem list at [24]). Cleary and Taback [8] have shown that Thompson's group F is not almost convex (in fact, Belk and Bux [1] have shown that F is not even minimally almost convex). Thus F is a potential example of an algorithmically stackable group that has none of these other algorithmic and geometric properties.

On the other hand, in [4], the present authors show that stackable groups are tame combable, as defined by Mihalik and Tschantz [21]. Tschantz [25] has conjectured that there exists a finitely presented group that is not tame combable. Such a group also would not admit the stackable property with respect to any finite symmetric generating set.

2. Notation

Throughout this paper, let G be a group with a finite *symmetric* generating set; that is, such that the generating set A is closed under inversion. Throughout the paper we assume that no element of A represents the identity element of G.

Let $\rho : A^* \to G$ be the canonical monoid homomorphism. A set $\mathcal{N}$ of *normal forms* for G over A is a subset of A^* such that the restriction of the map ρ to $\mathcal{N}$ is a bijection. As in Section 1, the symbol y_g denotes the normal form for $g \in G$. By slight abuse of notation, we use the symbol y_w to denote the normal form for $\rho(w)$ whenever $w \in A^*$.

Let ϵ denote the identity of G, and let 1 denote the empty word in A^*. For a word $w \in A^*$, we write w^{-1} for the formal inverse of w in A^*. For words $v, w \in A^*$, we write $v = w$ if v and w are the same word in A^*, and write $v =_G w$ if v and w represent the same element of G; that is, if $\rho(v) = \rho(w)$.

Let Γ be the Cayley graph of G with respect to A, with path metric d. A word $w \in A^*$ is called *geodesic* if w labels a geodesic path in Γ. Whenever $x \in A^*$ and $a \in A$, we write $e_{x,a}$ to denote the directed edge $e_{g,a}$ where $g = \rho(x)$ is the element of G represented by x. Define four maps

$\alpha : \vec{E} \to \mathbb{Q}$ by $\alpha(e_{g,a}) := \frac{1}{2}(d(\epsilon, g) + d(\epsilon, ga))$,

$\lambda : \vec{P} \to A^*$ by $\lambda(p) :=$ the word labeling the path p,

$\iota : \vec{E} \to G$ by $\iota(e_{g,a}) := g$, and

$\delta : G \times A^* \to \vec{P}$ by $\delta(g, w) :=$ the path in Γ starting at g labeled by w.

Given a presentation $\mathcal{P} = \langle A \mid R \rangle$ for G, the presentation is *symmetrized* if the generating set A is symmetric and the set R of defining relations is closed under inversion and cyclic conjugation. Let X be the Cayley 2-complex corresponding to this presentation, whose 1-skeleton is $X^1 = \Gamma$. Let $E(X)$ denote the set of undirected edges of X; we consider the
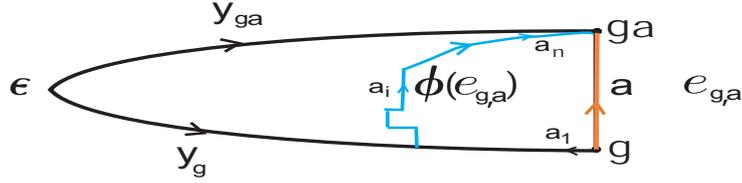
FIGURE 1. The stacking map

two directed edges $e_{g,a}$ and $e_{ga,a^{-1}}$ to have the same underlying directed edge in X between g and ga in X.

For an arbitrary word w in A^* that represents the trivial element ϵ of G, there is a *van Kampen diagram* Δ for w with respect to $\mathcal{P}$. That is, Δ is a finite, planar, contractible combinatorial 2-complex with edges directed and labeled by elements of A, satisfying the properties that the boundary of Δ is an edge path labeled by the word w starting at a basepoint vertex $*$ and reading counterclockwise, and every 2-cell in Δ has boundary labeled by an element of R. For any van Kampen diagram Δ with basepoint $*$, let $\pi_\Delta : \Delta \to X$ denote a cellular map such that $\pi_\Delta(*) = \epsilon$ and π_Δ maps edges to edges preserving both label and direction.

In general, there may be many different van Kampen diagrams for the word w. Also, we do not assume that van Kampen diagrams in this paper are reduced; that is, we allow adjacent 2-cells in Δ to be labeled by the same relator with opposite orientations.

See for example [3] or [20] for an exposition of the theory of van Kampen diagrams.

3. Procedures for constructing normal forms and van Kampen diagrams

The main goal of this section is to describe inductive procedures for finding normal forms and for constructing van Kampen diagrams for stackable groups. We begin with a discussion of the structure of the normal forms for a stackable group.

Let G be a group that is stackable over a symmetric generating set A, with stackable structure $(\mathcal{N}, <, k)$ from Definition 1.1. Then one can define a function $\phi : \vec{E}_r = \vec{E}_{r,\mathcal{N}} \to A^*$ by choosing, for each $e_{g,a} \in \vec{E}_r$, a label $\phi(e_{g,a}) = a_1 \cdots a_n \in A^*$ of a directed path in Γ satisfying the property that $\phi(e_{g,a}) =_G a$, $n \leq k$, and either $e_{ga_1 \cdots a_{i-1}, a_i} < e_{g,a}$ or $e_{ga_1 \cdots a_{i-1}, a_i} \in \vec{E}_d$ for each i. (Note that although we have $\phi(e_{g,a}) =_G a$, the fact that $<$ is a strict partial ordering implies that the word $\phi(e_{g,a})$ cannot be the letter a.) (See Figure 1.) This function is called a *stacking map*.

Let R_ϕ be the closure of the set of words $\{\phi(e_{g,a})a^{-1} \mid g \in G, a \in A\}$ under inversion, cyclic conjugation, and free reduction (except the empty word); R_ϕ is called the *stacking relation set*.

In the proof of the following proposition, we give an inductive procedure which, upon input of any word w over the generators of a stackable group, will output the normal form of the element $\rho(w)$ of the group G represented by w.

Proposition 3.1. *Let G be a stackable group. Then G is finitely presented, and the set of normal forms of the stackable structure is closed under taking prefixes.*

Proof. Let ϕ be a stacking map associated to a stackable structure on G over a finite generating set A, with normal form set $\mathcal{N}$, ordering $<$, and constant k. We begin by defining a relation $<_\phi$ on $\vec{E}$ as follows. Whenever e', e are both in $\vec{E}_r$ and e' lies in the path $\delta(\iota(e), \phi(e))$ in Γ starting at the initial vertex of e and labeled by $\phi(e)$ (and e' is oriented in the same direction as this path), write $e' <_\phi e$. Let $<_\phi$ be the transitive closure of this relation. Then $<_\phi$ is a subset of the well-founded strict partial ordering $<$ from Definition 1.1, and so is also a well-founded strict partial ordering. Moreover, the constant bound k on the lengths of words $\phi(e)$ together with König's Infinity Lemma imply that $<_\phi$ satisfies the property that for each $e \in \vec{E}_r$, there are only finitely many $e'' \in \vec{E}_r$ with $e'' <_\phi e$.

Using the stacking map, we describe a reduction procedure for finding the normal form for the group element associated to any word, by defining a rewriting operation on words over A, as follows.

Whenever a word $w \in A^*$ has a decomposition $w = xay$ such that $x, y \in A^*$, $a \in A$, and the directed edge $e_{x,a}$ of Γ lies in $\vec{E}_r$, then we rewrite $w \to x\phi(e_{x,a})y$. Now for every directed edge e' in the Cayley graph Γ that lies along the path $\delta(\rho(x), \phi(x))$, either e' is a degenerate edge in $\vec{E}_d$, or else $e' \in \vec{E}_r$ and $e' <_\phi e_{x,a}$. After rewriting a word w finitely many times $w \to w_1 \to \cdots \to w_i$, any decomposition $w_i = x_i a_i y_i$ with $x_i, y_i \in A^*$, $a_i \in A$, and $e_i := e_{x_i,a_i} \in \vec{E}_r$ satisfies the property that $e_i <_\phi e$ where $e = e_{x,a}$ for some decomposition $w = xay$ of the original word w. That is, each successive rewriting corresponds to one of the finitely many edges that are less (with respect to $<_\phi$) than the finitely many edges in the path labeled w in Γ starting at the identity vertex. Thus there can be at most finitely many rewritings $w \to w_1 \to \cdots \to w_m = z$ until a word z is obtained which cannot be rewritten with this procedure. The final step of the reduction procedure is to freely reduce the word z, resulting in a word w'.

Now $w =_G w'$, and the word w' (when input into this procedure) is not rewritten with the reduction procedure, since every prefix of w' is equal in G to a prefix of z. Write $w' = a_1 \cdots a_n$ with each $a_i \in A$. Then for all $1 \le i \le n$, the edge $e_i := e_{a_1 \cdots a_{i-1}, a_i}$ of Γ does not lie in $\vec{E}_r$, and so must be in $\vec{E}_d$. In the case that $i = 1$, this implies that one of the equalities of words $y_\epsilon a_1 = y_{a_1}$ or $y_{a_1} a_1^{-1} = y_\epsilon$ must hold (where ϵ denotes the identity of G). Now from Definition 1.1 we have that the normal form of the identity is the empty word, i.e. $y_\epsilon = 1$, and so the first equality $a_1 = y_{a_1}$ must hold. Assume inductively that $y_{a_1 \cdots a_i} = a_1 \cdots a_i$. The inclusion $e_{i+1} \in \vec{E}_d$ implies that either $a_1 \cdots a_i \cdot a_{i+1} = y_{a_1 \cdots a_{i+1}}$ or $y_{a_1 \cdots a_{i+1}} a_{i+1}^{-1} = a_1 \cdots a_i$. However, the latter equality on words would imply that the final letters on each side are the same, i.e. $a_{i+1}^{-1} = a_i$, which contradicts the fact that w' is freely reduced. Hence we have that $w' = y_{w'} = y_w$ is in normal form, and moreover every prefix of w' is also in normal form.

This reduction procedure uses only relators of the group lying in the stacking relation set R_ϕ to reduce any word $w \in A^*$ to its normal form. Hence R_ϕ is a set of defining relators

for G over the generating set A, and $\langle A \mid R_\phi \rangle$, is a presentation for G. Since the words in R_ϕ have length at most $k + 1$, the set R_ϕ and this presentation for G are also finite. $\quad\square$

We call $\langle A \mid R_\phi \rangle$ the *stacking presentation*. The prefix-closed set $\mathcal{N}$ of normal forms for a stackable group yields a maximal tree $\mathcal{T}$ in the Cayley graph Γ, namely the set of edges in the paths in Γ starting at ϵ and labeled by the words in $\mathcal{N}$. In the following Corollary we show that a stacking map yields a flow function associated to this tree.

Corollary 3.2. *A group G is stackable with respect to a finite symmetric generating set A if and only if the Cayley graph $\Gamma(G, A)$ admits a bounded flow function.*

Proof. First suppose that G is stackable over A, and let $\phi : \vec{E}_r \to A^*$ be a stacking map. From Proposition 3.1, the set $\mathcal{N}$ of normals forms from the stackable structure is prefix-closed; let $\mathcal{T}$ be the maximal tree in the Cayley graph $\Gamma = \Gamma(G, A)$ consisting of the edges lying in paths starting at ϵ labeled by words in $\mathcal{N}$. The set $\vec{E}_d$ of degenerate edges associated to the normal form set $\mathcal{N}$ is exactly the set of directed edges lying in this tree, and the edges of $\vec{E}_r$ are the edges of Γ that do not lie in the tree T.

Let $\Phi : \vec{E} \to \vec{P}$ be the function given by defining $\Phi(e_{g,a}) := e_{g,a}$ whenever $e_{g,a} \in \vec{E}_d$ and defining $\Phi(e_{g,a}) := \delta(g, \phi(e_{g,a}))$, the directed path in Γ with initial vertex g that is labeled by the word $\phi(e_{g,a})$, whenever $e_{g,a} \in \vec{E}_r$. That is, $\Phi|_{\vec{E}_d} = id_{\vec{E}_d}$ and $\Phi|_{\vec{E}_r} = \delta \circ (\iota \times \phi)$. Properties (F1), (F2d), and (F2r) of a flow function follow directly from the fact that ϕ is a stacking map. The constant k of the stackable structure is also a bound for this flow function.

Conversely, given a bounded flow function $\Phi : \vec{E} \to \vec{P}$ associated to a maximal tree $\mathcal{T}$ in the Cayley graph $\Gamma(G, A)$, let $\mathcal{N}$ be the set of normal forms for G over A consisting of the words labeling paths starting at ϵ that are geodesic (i.e. never backtrack) in $\mathcal{T}$. Let k be the constant bound on Φ, and let $<$ be the restriction of the ordering $<_\Phi$ to $\vec{E}_r$. Then $(\mathcal{N}, <, k)$ with the stacking map $\phi := \lambda \circ \Phi|_{\vec{E}_r}$ give a stackable structure for G over A. $\quad\square$

We note that the reduction procedure described in the proof of Proposition 3.1, for finding normal forms for words, may not be an algorithm. To make this process algorithmic, we would need to be able to recognize, given $x \in A^*$ and $a \in A$, whether or not $e_{x,a} \in \vec{E}_r$, and if so, be able to find $\phi(e_{x,a})$. That is, the set

$$\{(w, a, \phi(e_{w,a})) \mid e_{w,a} \in \vec{E}_r\} \cup \{(w, a, a) \mid e_{w,a} \in \vec{E}_d\}$$

should be computable (i.e., decidable or recursive). If we let Φ be the flow function associated to ϕ from Corollary 3.2, then (using the notation from Section 2) this set is the graph S_Φ of the function $A^* \times A \to A^*$ given by $(w, a) \mapsto \lambda(\Phi(e_{w,a}))$. In the case that S_Φ is computable, given any $(w, a) \in A^* \times A$, by enumerating the words z in A^* and checking in turn whether $(w, a, z) \in S_\Phi$, we can find $\lambda(\Phi(e_{w,a}))$. (Note that the set S_Φ is computable if and only if the set $\{(w, a, \phi(e_{w,a})) \mid w \in A^*, a \in A, e_{w,a} \in \vec{E}_r\}$ describing the graph of ϕ is computable. However, using the latter set in the reduction algorithm has the drawback of requiring us to enumerate the finite (and hence enumerable) set $\phi(\vec{E}_r)$, but we may not have an algorithm to find this set from the stackable structure.) Hence we have shown the following.
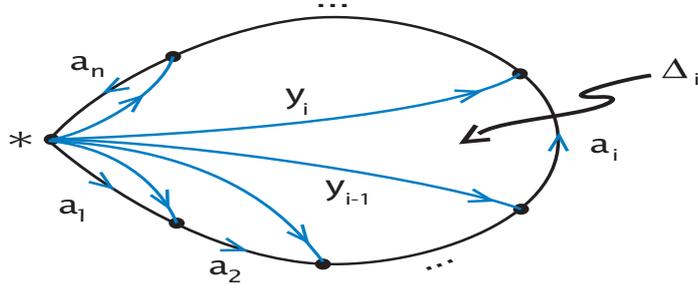
FIGURE 2. Van Kampen diagram built with seashell procedure

Theorem 3.3. *If G is algorithmically stackable, then G has solvable word problem.*

As with many other algorithmic classes of groups, it is natural to discuss formal language theoretic restrictions on the associated languages, and in particular on the set of normal forms. Computability of the set S_Φ implies that the set $\mathcal{N}$ is computable as well (since any word $a_1 \cdots a_n \in A^*$ lies in $\mathcal{N}$ if and only if the word is freely reduced and for each $1 \leq i \leq n$ the tuple $(a_1 \cdots a_{i-1}, a_i, a_i)$ lies in S_Φ). Many of the examples we consider in Section 4 will satisfy stronger restrictions on the set $\mathcal{N}$.

Next we turn to a discussion of building van Kampen diagrams in stackable groups. Before discussing the details of the inductive procedure for constructing these diagrams, we first reduce the set of diagrams required.

For a group G with symmetrized presentation $\mathcal{P} = \langle A \mid R \rangle$, a *filling* is a collection $\{\Delta_w \mid w \in A^*, w =_G \epsilon\}$ of van Kampen diagrams for all words representing the trivial element. Given a set $\mathcal{N} = \{y_g \mid g \in G\} \subseteq A^*$ of normal forms for G, a *normal form diagram* is a van Kampen diagram for a word of the form $y_g a y_{ga}^{-1}$ where $g \in G$ and a in A. We can associate this normal form diagram with the directed edge of the Cayley complex X labeled by a with initial vertex labeled by g. A *normal filling* for the pair $(G, \mathcal{P})$ consists of a set $\mathcal{N}$ normal forms for G that are *simple words* (i.e. words that label simple paths in the 1-skeleton of the Cayley complex X) including the empty word, together with a collection $\{\Delta_e \mid e \in E(X)\}$ of normal form diagrams, where for each undirected edge e in X, the normal form diagram Δ_e is associated to one of the two possible directions of e.

Every normal filling induces a filling, using the "seashell" ("cockleshell" in [2, Section 1.3]) method, illustrated in Figure 2, as follows. Given a word $w = a_1 \cdots a_n$ representing the identity of G, with each $a_i \in A$, then for each $1 \leq i \leq n$, there is a normal form diagram Δ_i in the normal filling that is associated to the edge of X with endpoints labeled by the group elements represented by the words $a_1 \cdots a_{i-1}$ and $a_1 \cdots a_i$. Letting y_i denote the normal form in $\mathcal{N}$ representing $a_1 \cdots a_i$, then the counterclockwise boundary of this diagram is labeled by either $y_{i-1} a_i y_i^{-1}$ or $y_i a_i^{-1} y_{i-1}^{-1}$; by replacing Δ_i by its mirror image if necessary, we may take Δ_i to have counterclockwise boundary word $x_i := y_{i-1} a_i y_i^{-1}$. We next iteratively build a van Kampen diagram Δ_i' for the word $y_\epsilon a_1 \cdots a_i y_i^{-1}$, beginning with $\Delta_1' := \Delta_1$. For $1 < i \leq n$, the planar diagrams Δ_{i-1}' and Δ_i have boundary subpaths

FIGURE 3. Van Kampen diagram Δ_e for degenerate edge e

sharing a common label y_i. The fact that this word y_i is simple, labeling a simple path in X, implies that the paths in the van Kampen diagrams Δ'_{i-1}, Δ_i labeled by y_i must also be simple, since the path in X is the image under the cellular maps $\pi_{\Delta'_{i-1}}$ and π_{Δ_i} of these boundary paths. Hence each of these boundary paths labeled y_i is an embedding in the respective van Kampen diagram. These paths are also oriented in the same direction, and so the diagrams Δ'_{i-1} and Δ_i can be glued, starting at their basepoints and folding along these subpaths, to construct the planar diagram Δ'_i. Performing these gluings consecutively for each i results in a van Kampen diagram Δ'_n with boundary label $y_\epsilon w y_w^{-1}$. Note that we have allowed the possibility that some of the boundary edges of Δ'_n may not lie on the boundary of a 2-cell in Δ'_n; some of the words x_i may freely reduce to the empty word, and the corresponding van Kampen diagrams Δ_i may have no 2-cells. Note also that the only simple word representing the identity of G is the empty word; that is, $y_\epsilon = y_w = 1$. Hence Δ'_n is the required van Kampen diagram for w.

Starting from a bounded flow function Φ for a stackable group G over a finite generating set A, we now describe the *stacking recursion*, which is an inductive procedure for constructing a filling for G over the stacking presentation $\mathcal{P} = \langle A \mid R_\phi \rangle$ by building a normal filling to which the seashell method can be applied, as follows. Let X be the Cayley complex of this presentation, and let $\mathcal{N}$ be the normal form set obtained from the maximal tree $\mathcal{T}$ associated to Φ. Since these normal forms label geodesics in a tree, each element w of $\mathcal{N}$ must be a simple word, i.e. labeling a simple path in X.

We define a normal form diagram corresponding to each directed edge in $\vec{E} = \vec{E}_r \cup \vec{E}_d$ of the Cayley graph as follows. Let e be an edge in $\vec{E}$, oriented from a vertex g to a vertex h and labeled by $a \in A$, and let $w_e := y_g a y_h^{-1}$.

In the case that e lies in $\vec{E}_d$, the word w_e freely reduces to the empty word. Let Δ_e be the van Kampen diagram for w_e consisting of a line segment of edges, with no 2-cells. (See Figure 3.)

In the case that $e \in \vec{E}_r$, we use Noetherian induction to construct the normal form diagram. Write $\lambda(\Phi(e)) = a_1 \cdots a_n$ with each $a_i \in A^*$, and for each $1 \leq i \leq n$, let e_i be the edge in the Cayley graph from $ga_1 \cdots a_{i-1}$ to $ga_1 \cdots a_i$ labeled by a_i. For each i, either the directed edge e_i is in $\vec{E}_d$, or else $e_i \in \vec{E}_r$ and $e_i <_\Phi e$; in both cases we have, by above or by Noetherian induction, a van Kampen diagram $\Delta_i := \Delta_{e_i}$ with boundary label $y_{ga_1 \cdots a_{i-1}} a_i y_{ga_1 \cdots a_i}^{-1}$. By using the "seashell" method, we successively glue the diagrams Δ_{i-1}, Δ_i along their common boundary words $y_{ga_1 \cdots a_{i-1}}$. Since all of these gluings are along simple paths, this results in a planar van Kampen diagram Δ'_e with boundary word $y_g \lambda(\Phi(e)) y_h^{-1}$. (Note that by our assumption that no generator represents the identity,
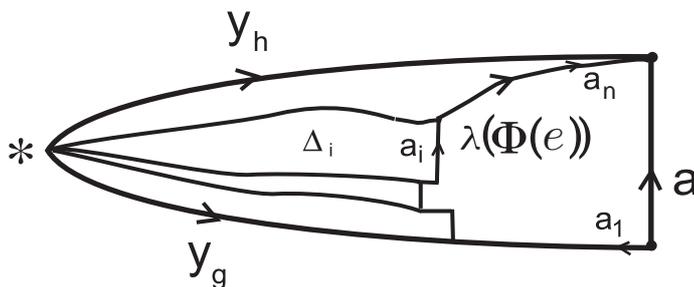
FIGURE 4. Van Kampen diagram Δ_e for recursive edge e

$\lambda(\Phi(e))$ must contain at least one letter.) Finally, glue a polygonal 2-cell with boundary label given by the relator $\lambda(\Phi(e))a^{-1}$ along the boundary subpath labeled $\lambda(\Phi(e))$ in Δ_e', in order to obtain the diagram Δ_e with boundary word w_e. Since in this step we have glued a disk onto Δ_e' along an arc, the diagram Δ_e is again planar, and is a normal form diagram corresponding to e. (See Figure 4.)

The penultimate step of the stacking recursion is to eliminate repetitions in order to obtain the normal filling associated to the flow function. Given any undirected edge e in $E(X)$, choose Δ_e to be a normal form diagram constructed above for one of the orientations of e. Then the collection $\mathcal{N}$ of normal forms, together with the collection $\{\Delta_e \mid e \in E(X)\}$ of normal form diagrams, is a normal filling for the stackable group G. Finally, we apply the seashell method again to obtain a filling.

Remark 3.4. The normal filling and filling constructed by the stacking recursion satisfy another useful property: *For every van Kampen diagram Δ in the filling and every vertex v in Δ, there is an edge path in Δ from the basepoint $*$ to v labeled by the normal form in $\mathcal{N}$ for the element $\pi_\Delta(v)$ in G.*

As with the earlier reduction procedure, we have an algorithm in the case that the set S_Φ is computable.

Proposition 3.5. *If G is algorithmically stackable over the finite generating set A, then the stacking recursion is an inductive algorithm which, upon input of a word $w \in A^*$ that represents the identity in G, will construct a van Kampen diagram for w over the stacking presentation.*

Although the reduction procedure (from the proof of Proposition 3.1) for finding normal forms for a stackable group can also be used to describe the van Kampen diagrams in this filling, it is this inductive view that connects more directly to the algorithms for solving the word problem and building van Kampen diagrams in the cases of almost convex groups and groups with finite complete rewriting systems.

Remark 3.6. For finitely generated groups that are not finitely presented, the concept of stackability can still be defined, although in this case it makes sense to discuss stacking maps in terms of a presentation for G, to avoid the (somewhat degenerate) case in which

every relator is included in the presentation. A group G with symmetrized presentation $\mathcal{P} = \langle A \mid R \rangle$ is $\mathcal{P}$-*stackable* if there is a bounded flow function Φ for a maximal tree in $\Gamma(G, A)$ satisfying the condition that the stacking relation set $R_{\lambda \circ \Phi}$ is a subset of R. Although we do not consider $\mathcal{P}$-stackable groups further in this paper, we note here that the reduction procedure for finding normal forms and the inductive method for constructing van Kampen diagrams over the presentation $\langle A \mid R_{\lambda \circ \Phi} \rangle$ of G (and hence over $\mathcal{P}$) described above still hold in this more general setting.

4. EXAMPLES OF STACKABLE GROUPS

4.1. **Illustration: Solvable Baumslag-Solitar groups.**

The solvable Baumslag-Solitar groups are presented by $G = BS(1, p) = \langle a, t \mid tat^{-1} = a^p \rangle$ with $p \in \mathbb{Z}$. A set of normal forms over the generating set $A = \{a, a^{-1}, t, t^{-1}\}$ is given by

$$\mathcal{N} := \{t^{-i} a^m t^k \mid i, k \in \mathbb{N} \cup \{0\}, \ m \in \mathbb{Z}, \ \text{and either } p \nmid m \text{ or } 0 \in \{i, k\}\}.$$

The recursive edges in $\vec{E}_r$ are the directed edges in the Cayley graph $\Gamma(G, A)$ of the form $e_{w,b}$ with initial vertex labeled w and edge label $b \in A$ satisfying one of the following:

(1) $w = t^{-i} a^m$ and $b = t^\eta$ with $m \neq 0$, $\eta \in \{\pm 1\}$, and $-i + \eta \leq 0$, or
(2) $w = t^{-i} a^m t^k$ and $b = a^\eta$ with $k > 0$ and $\eta \in \{\pm 1\}$.

We define a function $\phi : \vec{E}_r \to A^*$ by $\phi(e_{t^{-i} a^m, t^\eta}) := (a^{-\nu p} t a^\nu)^\eta$ in case (1), where $\nu := \frac{m}{|m|}$ is 1 if $m > 0$ and -1 if $m < 0$, and $\phi(e_{t^{-i} a^m t^k, a^\eta}) := t^{-1} a^{\eta p} t$ in case (2). Moreover let $\Phi : \vec{E} \to \vec{P}$ be defined by $\Phi|_{\vec{E}_d} = id_{\vec{E}_d}$ and $\Phi|_{\vec{E}_r} = \delta \circ (\iota \times \phi)|_{\vec{E}_r}$.

A portion of the Cayley graph and the corresponding function Φ in the case that $p = 2$ are illustrated in Figure 5; here thickened edges are degenerate, and the images $\Phi(e)$ for two recursive (dashed) edges e are shown with widely dashed paths. The double arrows within 2-cells indicate the direction of the flow through the Cayley complex toward the basepoint given by the function Φ.

Properties (F1) and (F2d) of the definition of a flow function, as well as boundedness, follow directly. To show that Φ also satisfies property (F2r), we first briefly describe the Cayley complex X for the finite presentation above; see for example [10, Section 7.4] for more details. The Cayley complex X is homeomorphic to the product $\mathbb{R} \times T$ of the real line with a regular tree T, and there are projections $\Pi_\mathbb{R} : X \to \mathbb{R}$ and $\Pi_T : X \to T$. The projection Π_T takes each edge labeled by an $a^{\pm 1}$ to a vertex of T. Each edge of T is the image of infinitely many t edges of the 1-skeleton X^1, with consistent orientation, and so we may consider the edges of T to be oriented and labeled by t, as well. For the normal form $y_g = t^{-i} a^m t^k \in \mathcal{N}$ of an element $g \in G$, the projection onto T of the path in X^1 starting at ϵ and labeled by y_g is the unique geodesic path, labeled by $t^{-i} t^k$, in the tree T from $\Pi_T(\epsilon)$ to $\Pi_T(g)$. For any directed edge e in $\vec{E}_r$ in case (2) above, there are $p + 1$ 2-cells in the Cayley complex X that contain e in their boundary, and the path $\Phi(e)$ is the portion of the boundary, disjoint from e, of the only one of those 2-cells σ that satisfies $d_T(\Pi_T(\epsilon), \Pi_T(q)) \leq d_T(\Pi_T(\epsilon), \Pi_T(e))$ for all points $q \in \sigma$, where d_T is the path metric in T.
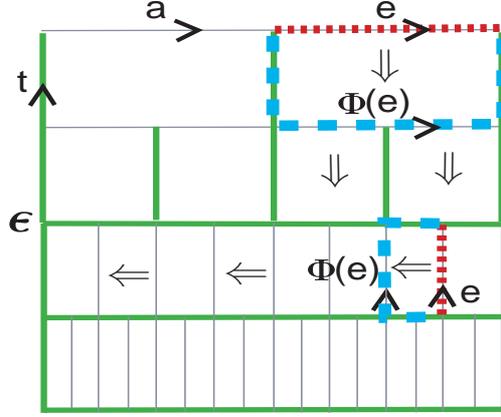
FIGURE 5. Flow function for $BS(1, 2)$

For any edge e' that lies both in this $\Phi(e)$ path and in $\vec{E}_r$, then e' is again a recursive edge of type (2), and we have $d_T(\Pi_T(\epsilon), \Pi_T(e')) < d_T(\Pi_T(\epsilon), \Pi_T(e))$. Thus the well-ordering on $\mathbb{N}$ applies, to show that the relation $<_\Phi$ is a well-founded strict partial ordering on the case (2) edges in $\vec{E}_r$.

The other projection map $\Pi_\mathbb{R}$ takes each vertex $t^{-i}a^m t^k$ to the real number $p^{-i}m$, and so takes each edge labeled by $t^{\pm 1}$ to a single real number, and takes each edge labeled $a^{\pm 1}$ to an interval in $\mathbb{R}$. For an edge $e \in \vec{E}_r$ in case (1) above, there are exactly two 2-cells in X containing e, and the path $\Phi(e)$ starting at the initial vertex $w = t^{-i}a^m$ of e travels around the boundary of the one of these two cells (except for the edge e) whose image, under the projection $\Pi_\mathbb{R}$, is closest to 0. The only possibly recursive edge e' in the path $\Phi(e)$ must also have type (1), and moreover the initial vertex of e' is $w' = t^{-i}a^{m-\nu}$ and satisfies $|\Pi_\mathbb{R}(w')| = |\Pi_\mathbb{R}(w)| - p^{-i}$. Then in all cases the relation $<_\Phi$ is a well-founded strict partial ordering, completing the proof of property (F2r). Therefore the function Φ is a bounded flow function, and the symmetrization of the presentation above is the stacking presentation.

4.2. Groups admitting complete rewriting systems.

A *finite complete rewriting system* (finite *CRS*) for a group G consists of a finite set A and a finite set of "rules" $R \subseteq A^* \times A^*$ (with each $(u, v) \in R$ written $u \to v$) such that as a monoid, G is presented by $G = Mon\langle A \mid u = v$ whenever $u \to v \in R \rangle$, and the rewritings $xuy \to xvy$ for all $x, y \in A^*$ and $u \to v$ in R satisfy: (1) Each $g \in G$ is represented by exactly one *irreducible* word (i.e. word that cannot be rewritten) over A, and (2) the relation on A^* defined by $x > y$ whenever $x \to x_1 \to ... \to x_n \to y$ is a well-founded strict partial ordering. (That is, there is no infinite chain $w \to x_1 \to x_2 \to \cdots$ of rewritings.)

Given any finite CRS (A, R) for G, there is another finite CRS (A, R') for G with the same set of irreducible words such that the CRS is *minimal*. That is, for each $u \to v$ in R', the word v and all proper subwords of the word u are irreducible (see, for example, [22,

p. 56]). If there is a letter $a \in A$ with $a =_G 1$, then the rewriting $a \to 1$ must be an element of R', and for all other $u \to v \in R'$, the letter a cannot appear in the words u or v. Let A' be the set A with all letters representing the identity of G removed, and let R'' be the set R' with all rules of the form $a \to 1$ for $a \in A \setminus A'$ removed. Now (A', R'') is also a minimal finite CRS for G over A with the same set of irreducible words. Next let A'' be the closure of A' under inversion. For each letter $a \in A'' \setminus A'$, there is an irreducible word $z_a \in A^*$ with $a =_G z_a$. Let $R''' := R'' \cup \{a \to z_a \mid a \in A' \setminus A\}$. Then (A'', R''') is again a minimal finite CRS for G, and with the same set of irreducible normal forms as the original CRS (A, R). For the remainder of this paper, we will assume that all of our complete rewriting systems are minimal and have an inverse-closed alphabet that does not contain a representative of ϵ.

Given any word $w \in A^*$, we write $w \overset{*}{\to} w'$ if there is any sequence of rewritings $w = w_0 \to w_1 \to \cdots \to w_n = w'$ (including the possibility that $n = 0$ and $w' = w$). A *prefix rewriting* of w with respect to the complete rewriting system (A, R) is a sequence of rewritings $w = w_0 \to \cdots \to w_n = w'$, written $w \overset{p*}{\to} w'$, such that at each w_i, the shortest possible reducible prefix is rewritten to obtain w_{i+1}. When w_n is irreducible, the number n is the *prefix rewriting length* of w, denoted $prl(w)$.

In Theorem 4.1, we apply ideas developed in the construction of a 1-combing associated to a finite complete rewriting system by Hermiller and Meier in [16], in order to build a stackable structure from a finite CRS.

Theorem 4.1. *A group admitting a finite complete rewriting system is algorithmically stackable.*

Proof. Let $\mathcal{N} = \{y_g \mid g \in G\}$ be the set of irreducible words from a minimal finite CRS (A, R) for a group G. Then $\mathcal{N} = A^* \setminus \cup_{u \to v \in R} A^* u A^*$. Note that prefixes of irreducible words are also irreducible, and so $\mathcal{N}$ is a prefix-closed set of normal forms for G over A.

Let Γ be the Cayley graph for the pair (G, A). As usual, the directed edge $e_{g,a}$ in Γ with label a and initial vertex g lies in the set $\vec{E}_d$ of degenerate edges if and only if $y_g a y_{ga}^{-1}$ freely reduces to the empty word, which in turn holds if and only if the undirected edge underlying $e_{g,a}$ lies in the tree of edges in paths from ϵ labeled by words in $\mathcal{N}$.

Define a function $\Phi : \vec{E} \to \vec{P}$ as follows. On degenerate edges, $\Phi|_{\vec{E}_d} := id_{\vec{E}_d}$. Given a recursive edge $e_{g,a} \in \vec{E}_r$, the word $y_g a$ is reducible, and since y_g is irreducible, the shortest reducible prefix of $y_g a$ is the entire word. Minimality of the rewriting system R implies that there is a unique factorization $y_g = w\tilde{u}$ such that $\tilde{u}a$ is the left hand side of a unique rule $\tilde{u}a \to v$ in R; that is, $y_g a \to wv$ is a prefix rewriting. Then define $\Phi(e_{g,a}) := \delta(g, \tilde{u}^{-1}v)$.

Properties (F1) and (F2d) of the definition of flow function are immediate. To check property (F2r), we first let p be the path $\Phi(e_{g,a})$ above labeled $\lambda(\Phi(e_{g,a})) = \tilde{u}^{-1}v$ for $e_{g,a} \in \vec{E}_r$. Since the word $\tilde{u}$ is a suffix of the normal form y_g, then the edges in the path p that correspond to the letters in $\tilde{u}^{-1}$ all lie in the set $\vec{E}_d$ of degenerate edges. For each directed edge e' in the subpath of p labeled by v, either e' also lies in $\vec{E}_d$, or else $e' \in \vec{E}_r$ and there is a factorization $v = v_1 a' v_2$ so that e' is the directed edge along p corresponding to the label $a' \in A$. In the latter case, if we denote the initial vertex of e' by g', then the prefix rewriting sequence from $y_{g'} a' v_2$ to its irreducible form is a (proper) subsequence of the

prefix rewriting of $y_g a$. That is, if we define a function $prl : \vec{E}_r \to \mathbb{N}$ by $prl(e_{h,b}) := prl(y_h b)$ whenever $e_{h,b} \in \vec{E}_r$, we have $prl(e') < prl(e_{g,a})$. Hence the ordering $<_\Phi$ corresponding to our function $\Phi : \vec{E} \to \vec{P}$ satisfies the property that $e' <_\Phi e$ implies $prl(e') < prl(e)$, and the well-ordering property on $\mathbb{N}$ implies that $<_\Phi$ is a well-founded strict partial ordering. Thus (F2r) holds as well, and Φ is a flow function.

The image set $\lambda(\Phi(\vec{E}))$ is the set of words $A \cup \{\tilde{u}^{-1}v \mid \exists a \in A \text{ with } \tilde{u}a \to v \text{ in } R\}$. Thus boundedness of Φ follows from finiteness of the sets A and R of generators and rules in the rewriting system. Then Corollary 3.2 shows that G is stackable over the generating set A.

To determine whether a tuple (w, a, x) (where $w, x \in A^*$ and $a \in A$) lies in the associated set S_Φ, we begin by computing the normal forms y_w and y_{wa} from w and wa, using the rewriting rules of our finite system. Then $(w, a, x) \in S_\Phi$ if and only if either at least one of the words $y_w a$ and $y_{wa}a^{-1}$ is irreducible and $a = x$, or else both of the words $y_w a$ and $y_{wa}a^{-1}$ are reducible and there exist a factorization $y_w = z\tilde{u}$ for some $z \in A^*$ and a rule $\tilde{u}a \to v$ in R such that $x = \tilde{u}^{-1}v$. Since there are only finite many rules in R to check for such a decomposition of y_w, it follows that the set S_Φ is also computable, and so this stackable structure is algorithmic. □

4.3. Thompson's group F.

Thompson's group
$$F = \langle x_0, x_1 \mid [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \rangle$$
is the group of orientation-preserving piecewise linear homeomorphisms of the unit interval [0,1], satisfying that each linear piece has a slope of the form 2^i for some $i \in \mathbb{Z}$, and all breakpoints occur in the 2-adics. In [7], Cleary, Hermiller, Stein, and Taback effectively show that Thompson's group with the generating set $A = \{x_0^{\pm 1}, x_1^{\pm 1}\}$ is stackable, with stacking presentation given by the symmetrization of the presentation above. Moreover, in [7, Definition 4.3] they give an algorithm for computing the stacking map, which can be used to show that F is algorithmically stackable.

Although we will not repeat their proof here, we describe the normal form set $\mathcal{N}$ associated to the stackable structure constructed for Thompson's group in [7] in order to discuss its formal language theoretic properties. Given a word w over the generating set $A = \{x_0^{\pm 1}, x_1^{\pm 1}\}$, denote the number of occurrences in w of the letter x_0 minus the number of occurrences in w of the letter x_0^{-1} by $\sigma_{x_0}(w)$; that is, $\sigma_{x_0}(w)$ is the exponent sum for x_0. The authors of that paper show ([7, Observation 3.6(1)]) that the set

$$\mathcal{N} := \{w \in A^* \mid \text{for all } \eta \in \{\pm 1\}, \text{ the words } x_0^\eta x_0^{-\eta}, x_1^\eta x_1^{-\eta}, \text{ and } x_0^2 x_1^\eta \text{ are not}$$
$$\text{subwords of } w, \text{ and for all prefixes } w' \text{ of } w, \sigma_{x_0}(w') \le 0\},$$

is a set of normal forms for F. Moreover, each of these words labels a (6,0)-quasi-geodesic path in the Cayley complex X [7, Theorem 3.7].

This set $\mathcal{N}$ is the intersection of the regular language $A^* \setminus \cup_{u \in U} A^* u A^*$, where $U := \{x_0 x_0^{-1}, x_0^{-1} x_0, x_1 x_1^{-1}, x_1^{-1} x_1, x_0^2 x_1, x_0^2 x_1^{-1}\}$, with the language $L := \{w \in A^* \mid \text{for all prefixes } w' \text{ of } w, \sigma_{x_0}(w') \le 0\}$. We refer the reader to the text of Hopcroft and Ullman [18] for

definitions and results on context-free and regular languages we now use to analyze the set L. The language L can be recognized by a deterministic push-down automaton (PDA) which pushes an x_0^{-1} onto its stack whenever an x_0^{-1} is read, and pops an x_0^{-1} off of its stack whenever an x_0 is read. When $x_1^{\pm 1}$ is read, the PDA does nothing to the stack, and does not change its state. The PDA remains in its initial state unless an x_0 is read when the only symbol on the stack is the stack start symbol Z_0, in which case the PDA transitions to a fail state (at which it must then remain upon reading the remainder of the input word). Ultimately the PDA accepts a word whenever its final state is its initial state. Consequently, L is a deterministic context-free language. Since the intersection of a regular language with a deterministic context-free language is deterministic context-free, the set $\mathcal{N}$ is also a deterministic context-free language. That is, the following holds.

Proposition 4.2. *Thompson's group F is algorithmically stackable with respect to a deterministic context-free language of normal forms.*

In Section 4.2, the normal form set of the stackable structure for a group with a finite complete rewriting system, $\mathcal{N} = A^* \setminus \cup_{u \to v \in R} A^* u A^*$, is a regular language. In consideration of the open question [24] of whether or not Thompson's group F has a finite complete rewriting system, it would also be of interest to know whether or not Thompson's group F is stackable with respect to a regular language of normal forms.

4.4. Almost convex groups.

Let G be a group with an inverse-closed generating set A, and let $d = d_\Gamma$ be the path metric on the associated Cayley graph Γ. For $n \in \mathbb{N}$, define the sphere $S(n)$ of radius n to be the set of points in Γ a distance exactly n from the vertex labeled by the identity ϵ, and define the ball $B(n)$ of radius n to be the set of points in Γ whose path metric distance to ϵ is less than or equal to n.

Definition 4.3. [5] *A group G is* almost convex *with respect to the finite symmetric generating set A if there is a constant k such that for all $n \in \mathbb{N}$ and for all g, h in the sphere $S(n)$ satisfying $d_\Gamma(g, h) \leq 2$ (in the Cayley graph $\Gamma = \Gamma(G, A)$), there is a path inside the ball $B(n)$ from g to h of length no more than k.*

Cannon [5] showed that every group satisfying an almost convexity condition over a finite generating set is also finitely presented. Thiel [23] showed that almost convexity is a property that depends upon the finite generating set used.

In the proof of Theorem 4.4 below, we show that a pair (G, A) that is almost convex is algorithmically stackable. Moreover the class of almost convex groups is exactly the class of geodesically stackable groups, and this must hold with respect to the shortlex normal forms. Given a choice of total ordering on A, a word $z_g \in A^*$ is the *shortlex normal form* for $g \in G$ if $\rho(z_g) = g$ and whenever $w \in A^*$ with $w =_G z_g$, then either the word lengths (in A^*) satisfy $l(w) > l(z_g)$, or else $l(w) = l(z_g)$ and w is lexicographically greater than z_g with respect to the ordering on A.

Theorem 4.4. *Let G be a group with finite generating set A. The following are equivalent:*

(1) *The pair (G, A) is almost convex.*

(2) *The pair (G, A) is geodesically stackable.*
(3) *The pair (G, A) is geodesically algorithmically stackable with respect to shortlex normal forms.*

Proof. Suppose that the group G has a finite symmetric generating set A, and let Γ be the corresponding Cayley graph with path metric d. The implication $(3) \Rightarrow (2)$ is immediate.

(1) implies (3):

Suppose that the group G is almost convex with respect to A, with an almost convexity constant k. Let $\mathcal{N} = \{z_g \mid g \in G\}$ be the set of shortlex normal forms over A for G (with respect to any choice of total ordering of A). Define a relation $<_\alpha$ on the set $\vec{E}_r = \vec{E}_{r,\mathcal{N}}$ of recursive edges by $e' <_\alpha e$ whenever $\alpha(e') < \alpha(e)$, where $\alpha(e_{g,a}) := \frac{1}{2}(d(\epsilon, g) + d(\epsilon, ga))$ for all $e_{g,a} \in \vec{E}_r$; then $<_\alpha$ inherits the property of being a well-founded strict partial ordering from the usual ordering on $\mathbb{N}[\frac{1}{2}]$.

Define a function $\phi : \vec{E}_r \to A^*$ as follows. Let $e_{g,a}$ be any element of $\vec{E}_r$.

Case I. If $d(\epsilon, g) = d(\epsilon, ga) = n$, then the points g and ga lie in the same sphere, and almost convexity of (G, A) implies that there is a directed edge path in Γ from g to ga of length at most k that lies in the ball $B(n)$. In this case define $\phi(e_{g,a})$ to be the shortlex least word over A that labels a path in $B(n)$ from g to ga. For any edge $e' \in \vec{E}_r$ lying in the path $\delta(g, \phi(e_{g,a}))$ (starting at g and labeled by $\phi(e_{g,a})$), the midpoint p of e' lies in $B(n)$, and so at least one of the endpoints of e' must lie in $B(n-1)$. Then $\alpha(e') \leq n - \frac{1}{2} < n = \alpha(e_{g,a})$.

Case II. If $d(\epsilon, g) = n$ and $d(\epsilon, ga) = n + 1$, then we can write $z_{ga} =_{A^*} z_h b$ for some $h \in G$ and $b \in A$. Hence $g, h \in S(n)$ and $d_\Gamma(g, h) \leq 2$. In this case we define $\phi(e_{g,a}) := xb$ where x is the shortlex least word over A that labels a path in $B(n)$ from g to h. The path $\delta(g, \phi(e_{g,a}))$ has length at most $k + 1$, and the final edge in this path, labeled by b, is degenerate. Thus any recursive edge $e' \in \vec{E}_r$ in this path lies in $B(n)$, and we have $\alpha(e') \leq n - \frac{1}{2} < n + \frac{1}{2} = \alpha(e_{g,a})$ in this case.

Case III. If $d(\epsilon, g) = n+1$ and $d(\epsilon, ga) = n$, then $z_g =_{A^*} z_{g'} c$ for some $c \in A$ and $g' \in G$, and we define $\phi(e_{g,a}) := c^{-1} y$ where y is the shortlex least word, of length at most k, labeling a path in $B(n)$ from g' to ga. The initial edge of $\delta(g, \phi(e_{g,a}))$ labeled c^{-1} is degenerate, and as in Case II we have $\alpha(e') < \alpha(e_{g,a})$ for all recursive edges e' in this path.

In all cases there is a path of length at most $k+1$ satisfying the conditions of Definition 1.1 for the ordering $<_\alpha$ on $\vec{E}_r$, and ϕ is a stacking map. Since the words in $\mathcal{N}$ are geodesics, then G is geodesically stackable over A with respect to the shortlex normal forms.

We are left with showing computability for the subset

$$S_\Phi = \{(w, a, \phi(e_{w,a})) \mid w \in A^*,\ a \in A,\ e_{w,a} \in \vec{E}_r\} \cup \{(w, a, a) \mid w \in A^*,\ a \in A,\ e_{w,a} \in \vec{E}_d\}$$

of $A^* \times A \times A^*$. Suppose that (w, a, x) is any element of $A^* \times A \times A^*$. Cannon [5, Theorem 1.4] has shown that the word problem is solvable for G, and so by enumerating the words in A^* in increasing shortlex order, and checking whether each in turn is equal in G to w, we can find the shortlex normal form z_w for w. Similarly we compute z_{wa}. If the word $z_w a z_{wa}^{-1}$ freely reduces to 1, then the tuple (w, a, x) lies in S_Φ if and only if $x = a$.

Suppose on the other hand that the word $z_w a z_{wa}^{-1}$ does not freely reduce to 1. If (as in Case I above) the word lengths $l(z_w) = l(z_{wa})$ both equal a natural number n, then we enumerate the elements of the finite set $\cup_{i=0}^{k} A^i$ of words of length up to k in increasing shortlex order. For each word $y = a_1 \cdots a_m$ in this enumeration, with each $a_i \in A$, we use the word problem solution again to compute the word length $l_{y,i}$ of the normal form $z_{wa_1 \cdots a_i}$ for each $0 \leq i \leq m$. If each $l_{y,i} \leq n$, and equalities $l_{y,i} = n$ do not hold for two consecutive indices i, then (w, a, x) lies in S_Φ if and only if $x = y$ and we halt the enumeration; otherwise, we go on to check the next word in our enumeration. The argument for Cases II-III in which $l(z_w) = l(z_{wa}) \pm 1$ are similar.

Combining the algorithms in the previous two paragraphs, the set S_Φ is computable and the stackable structure above for G is algorithmic.

(2) implies (1):

Suppose that the group G is geodesically stackable over the generating set A with respect to a set $\mathcal{N}$ of (geodesic) normal forms, and let $\mathcal{T}$ be the corresponding tree of degenerate edges. Let $\phi : \vec{E}_r = \vec{E}_{r,\mathcal{N}} \to A^*$ be an associated stacking map and let $\Phi : \vec{E} \to \vec{P}$ be the corresponding bounded flow function from Corollary 3.2. Let $M := \max\{l(\phi(e)) \mid e \in \vec{E}_r\}$ and let $k := 2M^2 + 2$. Also let g, h be any two points in a sphere $S(n)$ with $d_\Gamma(g, h) \leq 2$.

If $d(g, h) = 1$, then $h =_G ga$ for some $a \in A$. Moreover, since all normal forms in $\mathcal{N}$ are geodesics, the edge $e_{g,a}$ from g to h must be recursive. Then the path $p := \Phi(e_{g,a})$ labeled $\phi(e_{g,a})$ of length $\leq M < k$ from g to h satisfies the property that for every edge e' in p, either $e' \in \vec{E}_d$ or else $e' \in \vec{E}_r$ with $e' <_\Phi e$. Whenever $e' \in \vec{E}_r$, then applying Definition 1.4 we have $\alpha(e') < \alpha(e_{g,a}) = n$, and so the edge e' must lie in $B(n)$. If needed we replace each subpath of p whose edges all lie in $\vec{E}_d$ by the shortest path in the tree $\mathcal{T}$ of degenerate edges between the same endpoints. The effect of this replacement can only shorten the path p, and all edges in the new path must lie in $B(n)$.

On the other hand, suppose that $d(g, h) = 2$, with $h = gab$ for some $a, b \in A$. If $d(\epsilon, ga) = n - 1$, then there is a path of length $2 \leq k$ from g to h lying inside $B(n)$, and if $d(\epsilon, ga) = n$, we can apply the previous paragraph twice to obtain a path of length at most $2M < k$ from g to h via ga. Finally consider the case that $d(\epsilon, ga) = n + 1$, and write the (geodesic) normal form for ga as $y_{ga} = y_{g'}a'$ where $g' \in S(n)$ and $a' \in A$. It suffices to show that there is a path in $B(n)$ from g to g' of length at most $M^2 < \frac{1}{2}k$, since a similar proof results in such a path from g' to h. If $g = g'$ we are done, so suppose that $g \neq g'$.

Now the edge $e_{g',a'}$ lies in the tree $\mathcal{T}$ defined by $\mathcal{N}$. Since the normal forms are geodesic, there can only be one directed edge in $\mathcal{T}$ ending at $ga = g'a'$ and starting at a point in $S(n)$, and so the edge $e_{g,a}$ from g to ga must be recursive. Each recursive edge e' in the path $p := \Phi(e_{g,a})$ satisfies $\alpha(e') < \alpha(e_{g,a}) = n + \frac{1}{2}$, and so both endpoints of e' lie in $B(n)$. Replace each recursive edge e' in the path p satisfying $\alpha(e') = n$ by the directed path $\Phi(e')$, to obtain a new directed path p' of length at most M^2 from g to ga. Now for every recursive edge e'' in the path p', we have $\alpha(e'') < n$, and so all of the recursive edges in the path p' lie in $B(n)$.

Next as above we replace each subpath of p' consisting solely of degenerate edges in Γ by the shortest path in the tree $\mathcal{T}$ between the same endpoints, resulting in another path p''

from g to ga all of whose recursive edges lie in $B(n)$. The path p'' must end with a path in the tree T from a point in $B(n)$ to the vertex ga, and therefore the last directed edge of this path is the edge $e_{g',a'}$. Let $\tilde{p}$ be the path p'' with this last edge removed. Then $\tilde{p}$ is a path from g to g' lying in $B(n)$ of length at most M^2, as required. $\qquad\square$

Remark 4.5. Cannon's word problem algorithm for almost convex groups, which we applied in the proof of Theorem 4.4, requires the use of an enumeration of a finite set of words over A, namely those that represent ϵ in G and have length at most $k+2$. As Cannon also points out [5, p. 199], although this set is indeed recursive, there may not be an algorithm to find this set, starting from (G, A) and the constant k.

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