Rewriting Systems and Geometric 3-Manifolds *

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Abstract: The fundamental groups of most (conjecturally, all) closed 3-manifolds with uniform geometries have finite complete rewriting systems. The fundamental groups of a large class of amalgams of circle bundles also have finite complete rewriting systems. The general case remains open.

1. Introduction

A finite complete rewriting system for a group is a finite presentation which solves the word problem by giving a procedure for reducing each word down to a normal form. For closed irreducible 3-manifolds, results of [6] show that if the fundamental group is infinite and has a finite complete rewriting system, then the group has a tame combing, so results of [12] show that the manifold has universal cover homeomorphic to $\boldsymbol{R}^3$. In this paper we point out that well-known properties of finite complete rewriting systems and well-known facts about geometric 3-manifolds combine to give the following. (See below for definitions.)

Theorem 1. *Suppose that M is a closed 3-manifold bearing one of Thurston's eight geometries. Suppose further that if M is hyperbolic, that M virtually fibers over a circle. Then $\pi_1(M)$ has a finite complete rewriting system.*

According to a conjecture of Thurston ([15], question 18), every closed hyperbolic 3-manifold obeys the last hypothesis.

We also exhibit a class of non-uniform geometric 3-manifolds whose fundamental groups have finite complete rewriting systems. In particular, suppose that M is a graph of circle bundles based on a graph Γ. We will call an edge of this graph a *loop* if it has the

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same initial and terminal vertex. We suppose that when all loops are removed, the resulting graph is a tree. Under certain conditions on the way the vertex manifolds are glued along their boundary tori, the fundamental group $\pi_1(M)$ has a finite complete rewriting system.

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2. Background and Definitions

Let G be a group with finite generating set A. We write A^* for the free monoid on A. Each element of A evaluates into G under the identity map and this extends to a unique monoid homomorphism of A^* onto G which we denote by $w \to \overline{w}$.

A *rewriting system* R over the set A is a subset of $A^* \times A^*$. We write a pair $(u, v) \in R$ as $u \to v$ and call this a *rewriting rule* or *replacement rule.* If $u \to v$ is a rewriting rule, then for any $xuy \in A^*$, we write $xuy \to xvy$.

We say a finite set $R = \{u_i \to v_i\}$ is a *finite complete rewriting system* for (G, A) if

1) The monoid presentation $\langle A \mid u_i = v_i \rangle$ is a presentation of the underlying monoid of G.
2) There is no word $w_0 \in A^*$ spawning an infinite sequence of rewritings, $w_0 \to w_1 \to w_2 \to \cdots$. Such a system is called *Noetherian.*
3) For each element $g \in G$ there is exactly one word $w \in A^*$ so that $g = \overline{w}$ and w contains no u_i as a substring (that is, w is *irreducible*).

We will say that G *has a finite complete rewriting system* if there is a generating set A for which there is a finite complete rewriting system for (G, A).

Given a proposed rewriting system for (G, A), we would like to be able to check that it performs as advertised. Suppose that we know that 1) is satisfied. It turns out that we can often proceed in a mechanical manner. Property 2) can often be checked by giving an ordering $\prec$ on A^*. It then suffices to show that $\prec$ is a well-founded ordering and that $\to$ is $\prec$-decreasing. That is, we would like to know that any non-empty subset of A^* has a $\prec$-minimum element and that whenever $u \to v$, $v \prec u$. This can often be done by means of *recursive path ordering.*

Definition 2.1. [1] *Let $>$ be a partial well-founded ordering on a set S. The recursive path ordering $>_{rpo}$ on S^* is defined recursively from the ordering on S as follows. Given $s_1, ..., s_m, t_1, ..., t_n \in S$, $s_1...s_m >_{rpo} t_1...t_n$ if and only if one of the following holds.*
1) $s_1 = t_1$ and $s_2...s_m >_{rpo} t_2...t_n$.
2) $s_1 > t_1$ and $s_1...s_m >_{rpo} t_2...t_n$.
3) $s_2...s_m \geq_{rpo} t_1...t_n$.
The recursion is started from the ordering $>$ on S.

Recursive path ordering is a well-founded ordering which is compatible with concatenation of words [1]. Thus, in order to check that a rewriting system $R = \{u_i \to v_i\}$ is Noetherian, it suffices to order the generators so that $u_i >_{rpo} v_i$ for each $u_i \to v_i \in R$.

The critical pair analysis of the Knuth-Bendix algorithm [10] is a computational procedure for checking that a Noetherian rewriting system is complete. This algorithm checks for overlapping pairs of rules either of the form $r_1r_2 \to s, r_2r_3 \to t \in R$ with $r_2 \neq 1$, or of the form $r_1r_2r_3 \to s, r_2 \to t \in R$, where each $r_i \in A^*$; these are called *critical pairs.* In the first case, the word $r_1r_2r_3$ rewrites to both sr_3 and r_1t; in the second, it rewrites to both s and r_1tr_3. If there is a word $z \in A^*$ so that sr_3 and r_1t both rewrite to z in a finite number of steps in the first case, or so that s and r_1tr_3 both rewrite to z in the second case, then the critical pair is said to be *resolved*. The Knuth-Bendix algorithm checks that all of the critical pairs of the system are resolved; if this is the case, then the rewriting system is complete.

The existence of software packages implementing these procedures makes experimentation and exploration much less painful. In the course of our work we have used Rewrite Rule Laboratory [9], a software package for performing the Knuth-Bendix algorithm, as an aid in exploration.

3. Proof of Theorem 1

We will need the following facts about finite complete rewriting systems.

Proposition 3.0. *The trivial group has a finite complete rewriting system.*

Proposition 3.1. *$\mathbb{Z}$ has a finite complete rewriting system.*

Proposition 3.2. [5],[11] *If G is a surface group, then G has a finite complete rewriting system.*

Proposition 3.3. [3] *If H is finite index in G and H has a finite complete rewriting system, then G has a finite complete rewriting system.*

Proposition 3.4. [2] *If $1 \to K \to G \to Q \to 1$ is a short exact sequence, and K and Q have finite complete rewriting systems, then G has a finite complete rewriting system.*

A thorough account of Thurston's eight geometries is given in [14]. If M is a Riemannian manifold, then the Riemannian metric on M lifts to a Riemannian metric on the universal cover, $\widetilde{M}$. Suppose now that M is a closed 3-manifold with a uniform Riemannian metric. (This means that the isometry group of $\widetilde{M}$ acts transitively.) Thurston has shown that up to scaling, there are only eight possibilities for the Riemannian manifold $\widetilde{M}$ and that $\pi_1(M)$ is constrained in the following way. (We say that G is *virtually* H if G contains a finite index copy of H.)

Proposition 3.5. *Suppose M is a closed Riemannian 3-manifold with a uniform metric. Then one of the following holds:*

1) *$\widetilde{M}$ is the 3-sphere and $\pi_1(M)$ is finite, i.e., virtually trivial.*
2) *$\widetilde{M}$ is Euclidean 3-space and $\pi_1(M)$ is virtually $\mathbb{Z}^3$.*
3) *$\widetilde{M}$ is $S^2 \times \boldsymbol{R}$ and $\pi_1(M)$ is virtually $\mathbb{Z}$.*
4) *$\widetilde{M}$ is $\boldsymbol{H}^2 \times \boldsymbol{R}$ and $\pi_1(M)$ is virtually $H \times \mathbb{Z}$, where H is the fundamental group of a closed hyperbolic surface.*
5) *$\widetilde{M}$ is Nil, the Lie group consisting of upper triangular real 3×3 matrices with one's on the diagonal, and $\pi_1(M)$ contains a finite index subgroup G which sits in a short exact sequence $1 \to \mathbb{Z} \to G \to \mathbb{Z}^2 \to 1$.*

6) $\widetilde{M}$ *is* $\widetilde{\mathrm{PSL}_2(\mathbf{R})}$*, the universal cover of the unit tangent bundle of the hyperbolic plane, and* $\pi_1(M)$ *contains a finite index subgroup* G *which sits in a short exact sequence* $1 \to \mathbb{Z} \to G \to H \to 1$*, where* H *is the fundamental group of a closed hyperbolic surface.*

7) $\widetilde{M}$ *is Sol, a Lie group which is a semi-direct product of* $\boldsymbol{R}^2$ *with* $\boldsymbol{R}$*, and* $\pi_1(M)$ *has a finite index group* G *which sits in a short exact sequence* $1 \to \mathbb{Z}^2 \to G \to \mathbb{Z} \to 1$*.*

8) $\widetilde{M}$ *is hyperbolic space. Under the further assumption that* M *virtually fibers over a circle,* $\pi_1(M)$ *has a finite index group* G *which sits in a short exact sequence* $1 \to H \to G \to \mathbb{Z} \to 1$*, where* H *is the fundamental group of a closed hyperbolic surface.*

The proof of Theorem 1 now consists of applying Propositions 3.1 – 3.4 to the cases of Proposition 3.5. □

4. Non-uniform geometric 3-manifolds

For an arbitrary closed 3-manifold M satisfying Thurston's geometrization conjecture (see [14] for details), but not necessarily admitting a uniform Riemannian metric, finding rewriting systems becomes much more complicated. If M is not orientable, then M has an orientable double cover; Proposition 3.3 then says that if the fundamental group of the cover has a finite complete rewriting system, then so does M. So we may assume M is orientable.

Any closed orientable 3-manifold M can be decomposed as a connected sum $M = M_1 \# M_2 \# \cdots \# M_n$ in which each M_i is either a closed irreducible 3-manifold, or is homeomorphic to $S^2 \times S^1$ [4]. The fundamental group of M, then, can be written as the free product $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2) * \cdots * \pi_1(M_n)$. Another result of [2] says that the class of groups with finite complete rewriting systems is closed under free products; therefore, if $\pi_1(M_i)$ has a finite complete rewriting system for each i, then so does $\pi_1(M)$. So we may assume that our closed orientable 3-manifold is also irreducible.

Results of [7] and [8] state that a closed irreducible 3-manifold M can also be decomposed in a canonical way. (Much of this has been greatly simplified by [13].) There is a finite graph Γ associated to M. For each vertex of v of Γ, there is a compact 3-manifold $M_v \subset M$. For each edge e of Γ there is an incompressible torus $T_e^2 \subset M$. The boundary of M_v consists of $\coprod_{v \in \partial e} T_e^2$ and M is the union along these boundary tori of the pieces M_v. Consequently, the fundamental group of M can also be realized as the group of the graph of groups given by placing the fundamental groups of the vertex manifolds at the corresponding vertices of Γ, and the fundamental group of a torus on each edge, together with the appropriate injections.

If M satisfies Thurston's geometrization conjecture, then the interior of each one of these vertex manifolds admits a uniform Riemannian metric. The simplest case of this type of decomposition occurs when the graph Γ consists of a single vertex with no edges; this is dealt with in Theorem 1. In the case where Γ has edges, the vertex manifolds are either cusped hyperbolic 3-manifolds or Seifert fibered manifolds with boundaries.

In Section 5 we begin our pursuit of rewriting systems for non-uniform 3-manifolds. In this case our results depend on assuming that the vertex manifolds are circle bundles, that the graph obeys some moderate assumptions and that the gluings are of a particular type. In spite of these assumptions, the class of manifolds for which we can produce finite complete rewriting systems seems quite large. Its simplest cases are those in which Γ consists of a single edge with two vertices or one vertex. In these cases, $\pi_1(M)$ is a free product with amalgamation or an HNN extension. We begin with the case of a free product with amalgamation since this case illustrates the assumptions in our more general result.

5. Gluing two circle bundles

In this section we deal with the case in which M can be constructed by gluing two circle bundles together along a torus boundary. Suppose Γ is a graph with two vertices v and w and a single edge between them. The manifold M_v attached to the vertex v is formed by taking the product of a surface of genus g that has a single boundary component, with a circle. The fundamental group of the surface can be presented as

$$\left\langle a_1, ..., a_g, b_1, ..., b_g, p \mid p = \prod_{i=1}^{g} [a_i, b_i] \right\rangle = \langle a_1, ..., a_g, b_1, ..., b_g \rangle,$$

and the fundamental group of the circle as $\langle x \rangle$. Then the fundamental group A of M_v is

$$A = \pi_1(M_v) = \langle a_1, ..., a_g, b_1, ..., b_g \rangle \times \langle x \rangle.$$

The manifold M_w is also the product of a surface of genus h that has a single boundary component, with a circle, so we can write

$$C = \pi_1(M_w) = \langle c_1, ..., c_h, d_1, ..., d_h \rangle \times \langle y \rangle.$$

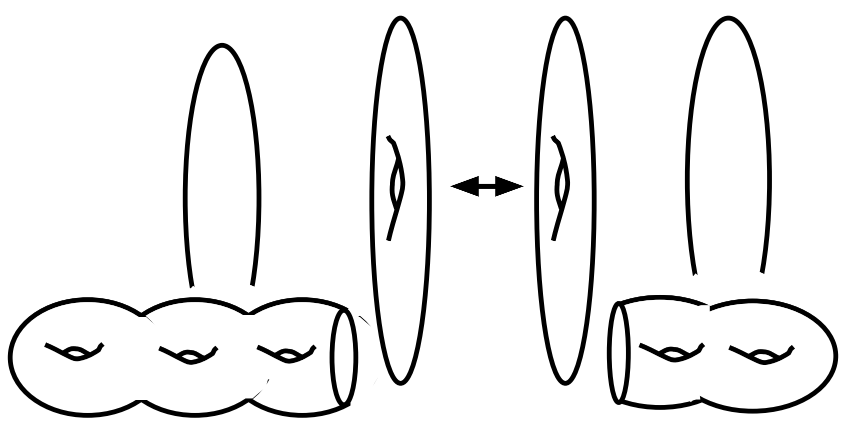

FIG. 1. The manifolds M_v and M_w

In each of these manifolds the fiber is represented schematically by a loop. The product of these loops with the surface boundaries gives each of these manifolds a torus boundary. These torus boundaries are glued together producing a single torus in the amalgam.

M_v and M_w are circle bundles, each with a single torus boundary component, and M is formed by gluing along these tori. The fundamental group G of M is then the free product with amalgamation of the fundamental groups of M_v and M_w, where the amalgamating subgroup is the fundamental group of the torus. So $G = \pi_1(M) = A *_X C$, where the subgroup $X \cong \mathbb{Z}^2$ is identified with the subgroups

$$\left\langle \prod_{i=1}^{g} [a_i, b_i] \right\rangle \times \langle x \rangle \quad \text{and} \quad \left\langle \prod_{i=1}^{h} [c_i, d_i] \right\rangle \times \langle y \rangle.$$

With this notation in hand, we see that each gluing is described by

$$x = \prod_{i=1}^{h} [c_i, d_i]^{k_1} y^{n_1} \quad \text{and} \quad \prod_{i=1}^{g} [a_i, b_i] = \prod_{i=1}^{h} [c_i, d_i]^{k_2} y^{n_2},$$

where the matrix

$$\phi = \begin{pmatrix} k_1 & n_1 \\ k_2 & n_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

If the gluing along X glues fibers of M_v to fibers of M_w, then M is itself a Seifert fibered manifold and consequently has a uniform geometry. The simplest nontrivial gluing is given by gluing the fiber of M_v to the surface boundary component of M_w and vice versa, giving

$$x = \prod_{i=1}^{h} [c_i, d_i] \quad \text{and} \quad \prod_{i=1}^{g} [a_i, b_i] = y.$$

In this case ϕ is the identity matrix. We have been able to find finite complete rewriting systems for a one parameter family of gluings, namely those where

$$\phi = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

so that

$$x = \prod_{i=1}^{h} [c_i, d_i] y^n \quad \text{and} \quad \prod_{i=1}^{g} [a_i, b_i] = y.$$

We will call gluings of this form *shear gluings.* For the case in which the gluing is a shear gluing, the fundamental group of the resulting manifold M has a presentation given by

$$G = \pi_1(M) = \Big\langle a_1, ..., a_g, b_1, ..., b_g, c_1, ..., c_h, d_1, ..., d_h, x, y \mid [a_i, x] = [b_i, x] = [c_j, y] = [d_j, y] = 1 \text{ (for all } i, j) \quad x = \prod_{i=1}^{h} [c_i, d_i] y^n, \ \prod_{i=1}^{g} [a_i, b_i] = y \Big\rangle.$$

Theorem 2. *If the manifold M can be decomposed into two vertex manifolds, such that each vertex manifold is the product of a surface with boundary with a circle, and such that these are connected with a shear gluing, then $\pi_1(M)$ has a finite complete rewriting system.*

Proof. We will utilize the presentation developed above for the fundamental group of M. The generating set for $G = \pi_1(M)$ will be $S \cup S^{-1}$ where

$$S = \{a_1, ..., a_g, b_1, ..., b_g, x, c_1, ..., c_h, d_1, ..., d_h, y\}.$$

The set R of rewriting rules is given by:

- inverse cancellation relators: $\{ss^{-1} \to 1 \quad s^{-1}s \to 1 \mid s \in S\}$
- blue vertex relators: $\{x^{\pm 1}a_i^{\pm 1} \to a_i^{\pm 1}x^{\pm 1}, \qquad x^{\pm 1}b_i^{\pm 1} \to b_i^{\pm 1}x^{\pm 1}\}$,
- red vertex relators: $\{c_i^{\pm 1}y^{\pm 1} \to y^{\pm 1}c_i^{\pm 1}, \qquad d_i^{\pm 1}y^{\pm 1} \to y^{\pm 1}d_i^{\pm 1}\}$,
- edge relators: $\{x^{\pm 1}y^{\pm 1} \to y^{\pm 1}x^{\pm 1}\}$
- v amalgam relators:
  $\{a_1b_1 \to yp^{-1}b_1a_1, \quad a_1b_1^{-1} \to b_1^{-1}py^{-1}a_1,$
  $a_1^{-1}yp^{-1}b_1 \to b_1a_1^{-1}, \quad a_1^{-1}b_1^{-1} \to b_1^{-1}a_1^{-1}yp^{-1}\}$,
- w amalgam relators:
  $\{c_1d_1 \to y^{-n}xq^{-1}d_1c_1, \quad c_1d_1^{-1} \to y^nd_1^{-1}qx^{-1}c_1,$
  $c_1^{-1}xq^{-1}d_1 \to y^nd_1c_1^{-1}, \quad c_1^{-1}d_1^{-1} \to y^nd_1^{-1}c_1^{-1}xq^{-1}\}$

Here we use the letter p to denote the string of letters

$$a_2b_2a_2^{-1}b_2^{-1}...a_gb_ga_g^{-1}b_g^{-1},$$

so p^{-1} denotes

$$b_ga_gb_g^{-1}a_g^{-1}...b_2a_2b_2^{-1}a_2^{-1}.$$

Similarly the letter q denotes the string

$$c_2d_2c_2^{-1}d_2^{-1}...c_hd_hc_h^{-1}d_h^{-1},$$

and q^{-1} denotes

$$d_hc_hd_h^{-1}c_h^{-1}...d_2c_2d_2^{-1}c_2^{-1}.$$

The discussion above Theorem 2 shows that these rewriting rules are a presentation of G, so this is a rewriting system for G.

In order to show that this system is Noetherian, we will show that there is a well-founded ordering on the words in $(S \cup S^{-1})^*$ so that whenever a word is rewritten, the resulting word is smaller with respect to this order. It suffices to take the appropriate order on S and use recursive path ordering.

The following lemma is proved by inspection of the rules in R.

Lemma 5.1. *Let $>$ be the recursive path ordering on $(S \cup S^{-1})^*$ induced by $a_1^{-1} > a_1 > b_1^{-1} > b_1 > \cdots > a_g^{-1} > a_g > b_g^{-1} > b_g > x > c_1^{-1} > c_1 > d_1^{-1} > d_1 > \cdots > c_h^{-1} > c_h > d_h^{-1} > d_h > y$. Then for each of the rules $u \to v$ in R, we have $u > v$.* □

It follows from Lemma 5.1 that this system is Noetherian. In order to show that the system is also complete, it suffices to show that in the monoid presented by R, for each element m in this monoid, there is exactly one word in $(S \cup S^{-1})^*$ representing m that cannot be rewritten.

We have used the critical pair analysis of the Knuth-Bendix procedure to check that the rewriting system R is complete. Rather than give the details of this computation, we will give a description of the normal forms that these rewriting rules produce.

To understand these normal forms, we consider several sublanguages.

Let $L(A/X)$ be the set of irreducible words on $\{a_i, b_i, y\}^{\pm 1}$ which do not end in $y^{\pm 1}$. Similarly, we let $L(X\backslash C)$ be the set of irreducible words on $\{c_i, d_i, x\}^{\pm 1}$ which do not begin in $x^{\pm 1}$. We take $L(X) = \{y^m x^n \mid m, n \in \mathbb{Z}\}$.

Lemma 5.2.
1) $L(A/X)$ bijects to A/X.
2) $L(X\backslash C)$ bijects to $X\backslash C$.
3) $L(X)$ bijects to X.

Proof. Clearly $L(A/X)$ surjects to A/X, for the set of reduced words on these letters surjects to A, and deleting any trailing $y^{\pm 1}$ does not change the coset. Thus, to prove *1)* we must show that if $u, u' \in L(A/X)$ with $\overline{u}X = \overline{u'}X$ then $u = u'$. Here u and u' both evaluate into the free group on $\{a_i, b_i\}$, so in this case we have $\overline{uy^m} = \overline{u'}$ for some m. Observe that no rewriting rule has a left hand side consisting of letters of $\{a_i, b_i, y\}^{\pm 1}$ and ending in $y^{\pm 1}$ (other than free reduction). Thus, if u is irreducible, then so is uy^m for any m. If $m \neq 0$ we have two distinct irreducible words representing the same group element. However, it is not hard to carry out the Knuth-Bendix procedure on the set of rules evaluating into A. This ensures that for any element of A there is a unique irreducible word and thus u and u' are identical as required.

The proof of *2)* is similar. Once again it is easy to see that $L(X\backslash C)$ surjects to $X\backslash C$. Now we suppose that $X\overline{w} = X\overline{w'}$ with w and w' in $L(X\backslash C)$. We then have $\overline{y^m x^n w} = \overline{w'}$ for some m and n. Once again, these are both irreducible as there is no rewriting rule beginning in $y^m x^n$ that can be applied. Appealing to the Knuth-Bendix procedure in C forces $m = n = 0$ and thus $w = w'$ as required.

The proof of *3)* is immediate. ☐

Now observe that any irreducible word θ has the form

$$\theta = u_1 v_1 w_1 \dots u_k v_k w_k$$

where

- For each i, $v_i = y^{m_i} x^{n_i} \in L(X)$.
- For each i, u_i is a maximal subword lying in $L(A/X)$; that is, u_i is a subword of θ lying in $L(A/X)$, and there is no larger subword of θ containing u_i that is also contained in $L(A/X)$.
- For each i, w_i is a maximal subword lying in $L(X\backslash C)$.

The maximality of each u_i ensures that no v_i consists solely of $y^{\pm 1}$'s directly preceding v_{i+1}. Likewise the maximality of each w_i ensures that no v_i consists solely of $x^{\pm 1}$'s directly following v_{i-1}.

We call k the *length* of θ. Let L_k be the set of all irreducible words of length k. For each $g \in G$ the *AC-length* of g is the minimal k such that $g \in (AC)^k$. Let G_k be the set of all group elements of AC-length k. That is, $g \in G_k$ if k is minimal such that $g = A_1C_1 \dots A_kC_k$ with $A_i \in A$, $C_i \in C$.

Lemma 5.3. *L_k bijects to G_k.*

Proof. This is an induction on k. When $k = 0$, the only irreducible word of length 0 is the empty word, and the only element of G with AC-length 0 is the trivial element, so there is a bijection between L_0 and G_0.

We next check the case $k = 1$. The set L_1 surjects to G_1, since any element of A has the form u_1v_1' and each element of C has the form $v_1''w_1$. Multiplying these together and applying the replacement rules produces a word of the form $u_1v_1w_1$ as required.

We now check that the map from L_1 to G_1 is injective. Suppose $g \in G_1$ and $g = \overline{u_1v_1w_1}$. Notice that A/X bijects to AC/C. Thus g determines a coset gC in AC/C and thus a unique element of A/X. Consequently, g determines u_1.

On the other hand, having determined the coset representative u_1 of gC in AC/C, there is a unique $c \in C$ so that $g = \overline{u_1}c$ and this, in turn, determines v_1w_1.

We now assume by induction that L_k bijects to G_k and check that L_{k+1} bijects to G_{k+1}. First check that L_{k+1} surjects to G_{k+1}. Suppose $g \in G_{k+1}$. Then g has the form g_kh with $g_k \in G_k$, $h \in G_1$. We represent g_k by a word of L_k and h by a word of L_1. We concatenate these words and apply our rewriting rules. The resulting word θ lies in $\cup_{i=1}^{k+1} L_i$. Since $\cup_{i=1}^{k} L_i$ misses G_{k+1}, it follows that $\theta \in L_{k+1}$.

We must show that L_{k+1} injects to G_{k+1}. Notice that g_k and h are determined by g up to an element of X. Thus, if $g = g_kh = g_k'h'$ then there is $z \in X$ so that $g_k' = g_kz$ and $h' = z^{-1}h$. Suppose then that g is represented by two irreducible words

$$\theta = u_1v_1w_1 \dots u_kv_kw_ku_{k+1}v_{k+1}w_{k+1}$$
$$\theta' = u_1'v_1'w_1' \dots u_k'v_k'w_k'u_{k+1}'v_{k+1}'w_{k+1}'$$

We take g_k and g_k' to be the group elements represented by the L_k portions of θ and θ'. Thus h and h' are represented by the remaining portions of these two words.

Notice that if w_k ends in $x^{\pm 1}$ then u_{k+1} and v_{k+1} are both empty, for otherwise, the final $x^{\pm 1}$'s of w_k would have moved right through u_{k+1} and any $y^{\pm 1}$'s of v_{k+1}. This cannot happen, since each w_i was chosen to be maximal. In the same manner, we do not have w_k empty and v_k ending in $x^{\pm 1}$. The same argument applies to θ'. Suppose the element $z \in X$ is represented by the word $y^mx^n \in L(X)$. Then if the word $u_1v_1w_1 \dots u_kv_kw_ky^mx^n$ is reduced using the rules of the rewriting system, for the resulting irreducible word $u_1v_1w_1 \dots u_{k-1}v_{k-1}w_{k-1}\tilde{u}_k\tilde{v}_k\tilde{w}_k$, we have that either $\tilde{w}_k = w_kx^n$ or $\tilde{w}_k = w_k$ is empty and $\tilde{v}_k = v_kx^n$. Now this irreducible word represents the same element of G_k as $u_1'v_1'w_1' \dots u_k'v_k'w_k'$. Therefore our induction hypothesis says that $\tilde{v}_k = v_k'$ and $\tilde{w}_k = w_k'$. So either $\tilde{w}_k = w_kx^n = w_k'$ or else $\tilde{w}_k$ and w_k' are both empty and $\tilde{v}_k = v_kx^n = v_k'$. Since w_k' cannot end with $x^{\pm 1}$ and if w_k' is empty then v_k' cannot end with $x^{\pm 1}$, this shows that n must be zero. Consequently, $u_1v_1w_1 \dots u_kv_kw_k$ and $u_1'v_1'w_1' \dots u_k'v_k'w_k'$ differ by at most a power of y in G; that is, we have $z = \overline{y^m}$.

On the other hand, if u_{k+1} begins in $y^{\pm 1}$, we must have v_k and w_k empty, for any leading $y^{\pm 1}$'s of w_{k+1} would have had to move left through w_k and any $x^{\pm 1}$'s of v_k. Maximality of the words u_i does not allow this to happen. Also, we cannot have u_{k+1}

is empty and v_{k+1} beginning with $y^{\pm 1}$. It follows by a similar argument, then, that $u_{k+1}v_{k+1}w_{k+1}$ and $u'_{k+1}v'_{k+1}w'_{k+1}$ differ in G by at most a power of x; that is, $z = \overline{x^n}$. Since z is now both a power of x and a power of y, that power is plainly 0, so $g_k = g'_k$ and $h = h'$. By induction

$$u_1 v_1 w_1 \dots u_k v_k w_k = u'_1 v'_1 w'_1 \dots u'_k v'_k w'_k$$

and

$$u_{k+1}v_{k+1}w_{k+1} = u'_{k+1}v'_{k+1}w'_{k+1},$$

so $\theta = \theta'$ as required. □

Since $G = \coprod G_k$ and the language of irreducible words is $\coprod L_k$ it follows that the language of irreducible words is a normal form which bijects to G. This completes the proof of Theorem 2. □

6. A graph of circle bundles

It is natural to ask whether the rewriting system in Section 5 can be modified to give rewriting systems for HNN extensions and larger graphs of circle bundles. In this Section we offer such rewriting systems. More specifically, suppose that Γ is a finite graph, and suppose that at any vertex v the vertex manifold M_v is a circle bundle over a surface with boundary of genus g_v. Then M_v will have a torus boundary component for each boundary component in the base surface; the edges of the graph Γ determine how these boundary components will be glued together.

We impose the following restriction on the graph Γ. Recall that a *loop* is an edge with the same initial and terminal vertex. We assume that if all of the loops in the graph Γ are removed, the resulting graph is a tree.

It now follows the vertices of Γ can be colored alternately red and blue, so that each edge that is not a loop joins a blue vertex to a red vertex. We use this to impose restrictions on the gluings. We orient all of the nonloop edges in Γ by taking the blue vertex to be the initial vertex, so the red vertex is the terminal vertex. Let V be the set of vertices of Γ, let E be the set of edges in Γ that are not loops, and let L be the set of loops in Γ. For each edge $e \in E$, $\iota(e)$ will denote the initial (blue) vertex of e, and $\tau(e)$ will denote the terminal (red) vertex. For each edge in $l \in L$, $\iota(l) = \tau(l)$ and this may be either red or blue.

Suppose that v is a blue vertex. The vertex manifold M_v is the product of a surface with boundary of genus g_v with a circle, as before. For each edge $e \in E$ with $\iota(e) = v$, there is one boundary component in the surface. Since v is blue, there are no edges ending at v. For each loop l with $\iota(l) = v$, there are two corresponding boundary components in the surface. The fundamental group of the surface can therefore be written as

$$\left\langle a_{v1}, ..., a_{vg_v}, b_{v1}, ..., b_{vg_v}, \{p_e \mid e \in E, \iota(e) = v\}, \{r_l, s_l \mid l \in L, \iota(l) = v\} \;\middle|\; \prod_{\iota(e)=v} p_e \prod_{\iota(l)=v} r_l s_l = \prod_{j=1}^{g_v} [a_{vj}, b_{vj}] \right\rangle.$$

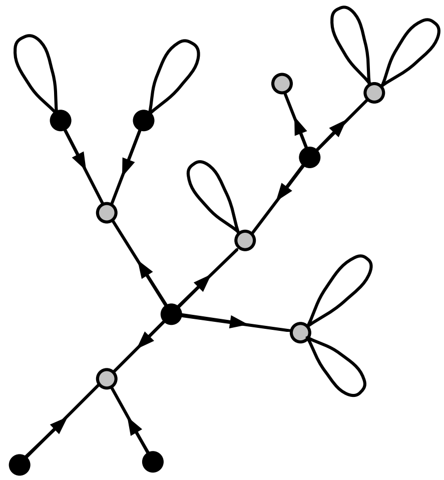

FIG. 2. A graph Γ satisfying our restrictions
The vertices are of two types, red and blue. After deleting loops, we are left with a tree. Each edge of this tree is directed from a blue vertex to a red vertex.

Then the fundamental group of the manifold is

$$\pi_1(M_v) = \left\langle a_{v1}, ..., a_{vg_v}, b_{v1}, ..., b_{vg_v}, \{p_e \mid e \in E, \iota(e) = v\}, \{r_l, s_l \mid l \in L, \iota(l) = v\} \;\middle|\; \prod_{\iota(e)=v} p_e \prod_{\iota(l)=v} r_l s_l = \prod_{j=1}^{g_v} [a_{vj}, b_{vj}] \right\rangle \times \langle x_v \rangle.$$

Suppose w is a red vertex; the vertex manifold M_w is constructed similarly as the product of a punctured surface of genus g_w with a circle. There are no edges that start at w, but each edge $e \in E$ with $\tau(e) = w$ corresponds to a single puncture in the surface, and each loop at w contributes two punctures. In this case we can present the fundamental group of M_w by

$$\pi_1(M_w) = \left\langle a_{w1}, ..., a_{wg_w}, b_{w1}, ..., b_{wg_w}, \{q_e \mid e \in E, \tau(e) = w\}, \{r_l, s_l \mid l \in L, \iota(l) = w\} \;\middle|\; \prod_{\tau(e)=w} q_e \prod_{\iota(l)=w} r_l s_l = \prod_{j=1}^{g_w} [a_{wj}, b_{wj}] \right\rangle \times \langle x_w \rangle.$$

Suppose the edge $e \in E$ has initial vertex v (so v is blue) and terminal vertex w (so w is red). Then the manifolds M_v and M_w are glued along the tori given by the product of corresponding punctures with the circle. On the level of fundamental groups, this gives an amalgamation with relations

$$x_v = q_e^{k_e} x_w^{n_e} \quad \text{and} \quad p_e = q_e^{k'_e} x_w^{n'_e}$$

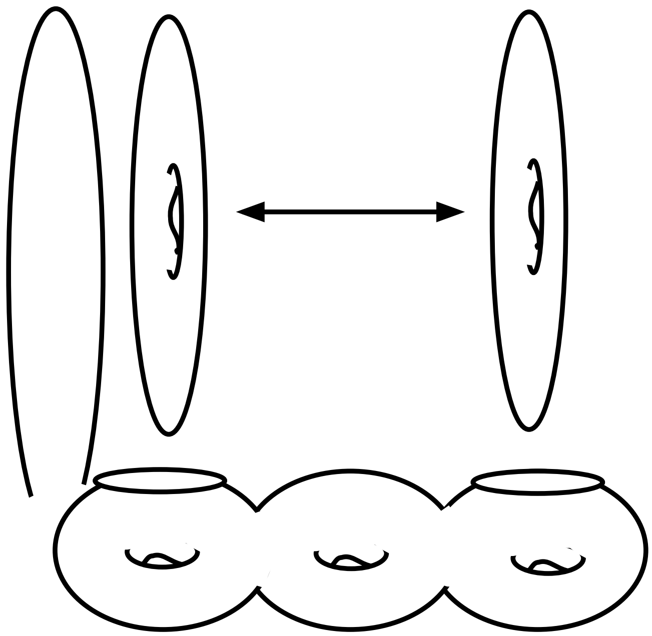

FIG. 3. A graph manifold with a single vertex manifold M_v

The surface has two boundary components. The fiber is represented schematically by a loop. The product of the loop with the surface's two boundary components gives M_v two torus boundary components. These torus boundary components are glued together producing a single torus in the graph manifold.

where the matrix

$$\phi = \begin{pmatrix} k_e & n_e \\ k'_e & n'_e \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

As before, we will call this a *shear gluing* if the matrix is of the form

$$\phi = \begin{pmatrix} 1 & n_e \\ 0 & 1 \end{pmatrix};$$

our relations in this case are

$$x_v = q_e x_w^{n_e} \quad \text{and} \quad p_e = x_w.$$

For each red vertex w, define the number n_w to be the sum over all the edges $e \in E$, with target $\tau(e) = w$, of the numbers n_e.

If the loop l has initial and terminal vertex z (z can be either red or blue), then a generator t_l is added to form the presentation for M, along with relations

$$t_l x_z t_l^{-1} = s_l^{k_l} x_z^{m_l} \quad \text{and} \quad t_l r_l t_l^{-1} = s_l^{k'_l} x_z^{m'_l}$$

where, again, the matrix

$$\phi = \begin{pmatrix} k_l & m_l \\ k'_l & m'_l \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

As before, in order to find a finite complete rewriting system, we assume this matrix is

$$\phi = \begin{pmatrix} 1 & m_l \\ 0 & 1 \end{pmatrix};$$

our relations in this case are

$$t_l x_z t_l^{-1} = s_l x_z^{m_l} \quad \text{and} \quad t_l r_l t_l^{-1} = x_z,$$

and we also refer to these as *shear gluings*.

The fundamental group for the manifold M, then, is the free product of the fundamental groups of the vertex manifolds, with the amalgamations for each edge, and the HNN extensions for each loop, described above. In order to simplify this presentation for M, for each edge $e \in E$, replace the generator p_e with the generator $x_{\tau(e)}$ in this presentation, and replace the generator q_e with the word $x_{\iota(e)} x_{\tau(e)}^{-n_e}$. Then the presentation for M, assuming the restriction on Γ and assuming that all of the gluings are shear, is given by

$$\begin{aligned}\pi_1(M) = \Big\langle a_{zj}, b_{zj}, x_z, r_l, s_l, t_l \;\Big|\; & \\ & [a_{zj}, x_z] = [b_{zj}, x_z] = [r_l, x_{\iota(l)}] = [s_l, x_{\iota(l)}] = [x_{\iota(e)}, x_{\tau(e)}] = 1, \\ & \prod_{\iota(e)=v} x_{\tau(e)} \prod_{\iota(l)=v} r_l s_l = \prod_{j=1}^{g_v} [a_{vj}, b_{vj}], \\ & \prod_{\tau(e)=w} x_{\iota(e)} x_w^{-n_e} \prod_{\tau(l)=w} r_l s_l = \prod_{j=1}^{g_w} [a_{wj}, b_{wj}], \\ & t_l x_{\iota(l)} t_l^{-1} = s_l x_{\iota(l)}^{m_l}, \quad t_l r_l t_l^{-1} = x_{\iota(l)} \Big\rangle,\end{aligned}$$

where the generators and relations range over all vertices z, all $1 \le j \le g_z$, all blue vertices v, all red vertices w, all edges e in E, and all loops l.

Theorem 3. *Suppose that Γ is a graph for which, when all of the loops in Γ are removed, the resulting graph is a tree. If the manifold M can be decomposed into a graph of circle bundles with graph Γ, such that each vertex manifold is the product of a surface with boundary with a circle, and such that the gluing corresponding to each edge and loop is a shear gluing, then $\pi_1(M)$ has a finite complete rewriting system.*

Proof. Using the presentation above, the following is a rewriting system for the graph of circle bundles M, with alphabet $A = S \cup S^{-1}$, where

$$S = \{a_{zj}, b_{zj}, x_z, r_l, s_l, t_l \mid z \in V, 1 \le j \le g_z, l \in L\}.$$

- inverse cancellation relators: $\{ss^{-1} \to 1, \quad s^{-1}s \to 1 \mid s \in S\}$
- blue vertex relators:
  $\{x_v^{\pm 1} a_{vi}^{\pm 1} \to a_{vi}^{\pm 1} x_v^{\pm 1}, \qquad x_v^{\pm 1} b_{vi}^{\pm 1} \to b_{vi}^{\pm 1} x_v^{\pm 1},$
  $x_{\iota(k)}^{\pm 1} r_k^{\pm 1} \to r_k^{\pm 1} x_{\iota(k)}^{\pm 1}, \qquad x_{\iota(k)}^{\pm 1} s_k^{\pm 1} \to s_k^{\pm 1} x_{\iota(k)}^{\pm 1}\}$

- red vertex relators:
  $\{a_{wi}^{\pm 1}x_w^{\pm 1} \to x_w^{\pm 1}a_{wi}^{\pm 1}, \qquad b_{wi}^{\pm 1}x_w^{\pm 1} \to x_w^{\pm 1}b_{wi}^{\pm 1},$
  $r_l^{\pm 1}x_{\iota(l)}^{\pm 1} \to x_{\iota(l)}^{\pm 1}r_l^{\pm 1}, \qquad s_l^{\pm 1}x_{\iota(l)}^{\pm 1} \to x_{\iota(l)}^{\pm 1}s_l^{\pm 1}\}$
- edge relators: $\{x_{\iota(e)}^{\pm 1}x_{\tau(e)}^{\pm 1} \to x_{\tau(e)}^{\pm 1}x_{\iota(e)}^{\pm 1}\}$
- blue amalgam relators:
  $\{a_{v1}b_{v1} \to \Lambda_v \prod_{j=g_v}^{2}[b_{vj}, a_{vj}]b_{v1}a_{v1},$
  $a_{v1}b_{v1}^{-1} \to b_{v1}^{-1}\prod_{j=2}^{g_v}[a_{vj}, b_{vj}]\Lambda_v^{-1}a_{v1},$
  $a_{v1}^{-1}\Lambda_v \prod_{j=g_v}^{2}[b_{vj}, a_{vj}]b_{v1} \to b_{v1}a_{v1}^{-1},$
  $a_{v1}^{-1}b_{v1}^{-1} \to b_{v1}^{-1}a_{v1}^{-1}\Lambda_v \prod_{j=g_v}^{2}[b_{vj}, a_{vj}]\}$
- red amalgam relators:
  $\{a_{w1}b_{w1} \to x_w^{-n_w}\Omega_w \prod_{j=g_w}^{2}[b_{wj}, a_{wj}]b_{w1}a_{w1},$
  $a_{w1}b_{w1}^{-1} \to x_w^{n_w}b_{w1}^{-1}\prod_{j=2}^{g_w}[a_{wj}, b_{wj}]\Omega_w^{-1}a_{w1},$
  $a_{w1}^{-1}\Omega_w \prod_{j=g_w}^{2}[b_{wj}, a_{wj}]b_{w1} \to x_w^{n_w}b_{w1}a_{w1}^{-1},$
  $a_{w1}^{-1}b_{w1}^{-1} \to x_w^{-n_w}b_{w1}^{-1}a_{w1}^{-1}\Omega_w \prod_{j=g_w}^{2}[b_{wj}, a_{wj}]\}$
- blue HNN relators:
  $\{x_{\iota(k)}t_k \to t_k r_k, \qquad x_{\iota(k)}^{-1}t_k \to t_k r_k^{-1},$
  $r_k t_k^{-1} \to t_k^{-1}x_{\iota(k)}, \qquad r_k^{-1}t_k^{-1} \to t_k^{-1}x_{\iota(k)}^{-1}$
  $s_k t_k \to t_k r_k^{-m_k}x_{\iota(k)}, \qquad s_k^{-1}t_k \to t_k r_k^{m_k}x_{\iota(k)}^{-1},$
  $x_{\iota(k)}t_k^{-1} \to t_k^{-1}s_k x_{\iota(k)}^{m_k}, \qquad x_{\iota(k)}^{-1}t_k^{-1} \to t_k^{-1}s_k^{-1}x_{\iota(k)}^{-m_k}\},$

  and
- red HNN relators:
  $\{t_l r_l \to x_{\iota(l)}t_l, \qquad t_l r_l^{-1} \to x_{\iota(l)}^{-1}t_l,$
  $t_l^{-1}x_{\iota(l)} \to r_l t_l^{-1}, \qquad t_l^{-1}x_{\iota(l)}^{-1} \to r_l^{-1}t_l^{-1},$
  $t_l x_{\iota(l)} \to x_{\iota(l)}^{m_l}s_l t_l, \qquad t_l x_{\iota(l)}^{-1} \to x_{\iota(l)}^{-m_l}s_l^{-1}t_l,$
  $t_l^{-1}s_l \to x_{\iota(l)}r_l^{-m_l}t_l^{-1}, \qquad t_l^{-1}s_l^{-1} \to x_{\iota(l)}^{-1}r_l^{m_l}t_l^{-1}\}$

These rules range over all blue vertices v, red vertices w, and edges $e \in E$, as well as all loops k at blue vertices and all loops l at red vertices. The letter Λ_v denotes the string of letters

$$\Lambda_v = \prod_{\iota(e)=v} x_{\tau(e)} \prod_{\iota(k)=v} r_k s_k.$$

Λ_v^{-1}, then, denotes the formal inverse of Λ_v, taking the letters in the string Λ_v in the opposite order with their signs changed. The letter Ω_w denotes the string of letters

$$\Omega_w = \prod_{\tau(e)=w} x_{\iota(e)} \prod_{\iota(l)=w} r_l s_l,$$

and Ω_w^{-1} is its formal inverse.

Denote this set of rules to be R; the discussion preceding Theorem 3 shows that the generators A together with our rewriting rules R give a presentation for the fundamental group of the graph of circle bundles.

In order to show that this rewriting system is complete, we will first show that a subset of the rules give rise to a complete rewriting system. Let

$$A' = A - \{t_k^{\pm 1} \mid k \in L, \iota(k) \text{ is blue}\},$$

and define R' to be the rewriting system consisting of all of the rules above except the blue HNN relators and the inverse cancellation relators involving the letters of $A - A'$.

In order to show that this system R' is Noetherian, we will again show that these rules decrease the well-founded recursive path ordering. The following lemma is proved by inspection of the rules in the set R'.

Lemma 6.1. *Let $>$ be the recursive path ordering induced by*

$$t_l^{-1} > t_l > a_{w1}^{-1} > a_{w1} > b_{w1}^{-1} > b_{w1} > a_{w2}^{-1} > \cdots > b_{wg_w} > x_v^{-1} > r_l^{-1} > r_l > s_l^{-1} > s_l >$$
$$x_v > a_{v1}^{-1} > a_{v1} > b_{v1}^{-1} > b_{v1} > a_{v2}^{-1} > \cdots > b_{vg_v} > r_k^{-1} > r_k > s_k^{-1} > s_k > x_w^{-1} > x_w,$$

where v is any blue vertex, w is any red vertex, k is any loop at a blue vertex, and l is any loop at a red vertex. Then for each of the rules $u \to v$ in R', we have $u > v$.

It follows from the Lemma that this system R' is Noetherian. In order to show that R' is also complete, the remaining property to check is that in the monoid presented by (A', R'), for each element m in this monoid, there is exactly one word in A'^* representing m that cannot be rewritten. This proof has been done in Section 5 when $\Gamma$is a graph with a single edge; for more complicated graphs, this becomes much more difficult. For the rewriting system R', we have checked that R' is complete using the Knuth-Bendix algorithm [10].

Since the rewriting system R' is complete, for each word $u \in A'^*$, there is a bound on the lengths of all sequences of rewritings $u \to w_1 \to \cdots \to w_n$ (where the length of this sequence is defined to be n). The maximum of the lengths of all of the possible rewritings of u is called the *disorder* of u, denoted $d_{R'}(u)$. We will use these numbers in order to show that the larger rewriting system R is Noetherian.

In order to define a well-founded ordering on the set A^*, note that every word $w \in A^*$ can be written uniquely in the form

$$w = u_1 t_1 u_2 t_2 \cdots u_j t_j u_{j+1},$$

where each u_i is a (possibly empty) word in A'^* and each t_i is a letter in $(A-A') \cup (A-A')^{-1}$. Define functions ψ_i from A^* to the nonnegative integers by

$$\begin{aligned} \psi_0(w) =& j, \\ \psi_{2i}(w) =& d_{R'}(u_i), \text{and} \\ \psi_{2i+1}(w) =& length(u_i), \end{aligned}$$

where i ranges from 1 to $j+1$, and *length* denotes the word length over A'. In order to compare words of different length, define $\psi_i(w) = 0$ if $i > 2j+3$. For two words w_1 and

w_2 in A^*, define $w_1 > w_2$ if $\psi_0(w_1) > \psi_0(w_2)$ or if $\psi_i(w_1) = \psi_i(w_2)$ for all $i < k$ and $\psi_k(w_1) > \psi_k(w_2)$. We claim this defines a well-founded ordering on A^*.

To check the claim, suppose $w \in A^*$. If a rule in R' is applied to w, the rule must to be applied to one of the subwords u_i, so the value of ψ_{2i} is reduced without altering the values of ψ_k for any $0 \leq k \leq 2i-1$. Suppose an inverse cancellation relator involving the letters of $A - A'$ is applied to w; in this case, the value of ψ_0 is reduced. Finally, if a blue HNN relator is applied to w, the rule must be applied to a subword $u_i t_i$ of w. Then the values of ψ_k for any $0 \leq k \leq 2i-1$ are not altered; the value of ψ_{2i} either decreases or remains unchanged; and the value of ψ_{2i+1} is reduced. So each time a word is rewritten, the resulting word is smaller with respect to this ordering. Therefore the rewriting system R is also Noetherian.

Since R is Noetherian, we have again applied the Knuth-Bendix procedure to check that the rewriting system R is complete. The dedicated reader may wish to check this; our computation resolved 84 critical pairs. □

7. A question

When the gluings of the circle bundles at the vertices of Γ are more complicated, or when the circle bundles themselves are replaced by more general Seifert-fibered spaces, we were unable to find finite complete rewriting systems. So we end with the following.

Question. *Does every fundamental group of a closed 3-manifold satisfying Thurston's geometrization conjecture have a finite complete rewriting system?*

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