Monomial orderings, rewriting systems, and Gröbner bases for the commutator ideal of a free algebra

Susan M. Hermiller
Department of Mathematics and Statistics
University of Nebraska-Lincoln
Lincoln, NE 68588
smh@math.unl.edu

Xenia H. Kramer
Department of Mathematics
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061
xenia@math.vt.edu

Reinhard C. Laubenbacher
Department of Mathematics
New Mexico State University
Las Cruces, NM 88003
reinhard@nmsu.edu

[0]*Key words and phrases.* non-commutative Gröbner bases, rewriting systems, term orderings.

1991 Mathematics Subject Classification. 13P10, 16D70, 20M05, 68Q42

The authors thank Derek Holt for helpful suggestions. The first author also wishes to thank the National Science Foundation for partial support.

Abstract

In this paper we consider a free associative algebra on three generators over an arbitrary field K. Given a term ordering on the commutative polynomial ring on three variables over K, we construct uncountably many liftings of this term ordering to a monomial ordering on the free associative algebra. These monomial orderings are total well orderings on the set of monomials, resulting in a set of normal forms. Then we show that the commutator ideal has an infinite reduced Gröbner basis with respect to these monomial orderings, and all initial ideals are distinct. Hence, the commutator ideal has at least uncountably many distinct reduced Gröbner bases. A Gröbner basis of the commutator ideal corresponds to a complete rewriting system for the free commutative monoid on three generators; our result also shows that this monoid has at least uncountably many distinct minimal complete rewriting systems.

The monomial orderings we use are not compatible with multiplication, but are sufficient to solve the ideal membership problem for a specific ideal, in this case the commutator ideal. We propose that it is fruitful to consider such more general monomial orderings in non-commutative Gröbner basis theory.

1 Introduction

Let K be a field, let $\Sigma = \{a, b, c\}$, and let $A = K\langle a, b, c\rangle$ be the free associative algebra over K on Σ. Let

$$\gamma : K\langle a, b, c\rangle \longrightarrow K[x, y, z]$$

be the projection modulo the commutator ideal

$$I = (ab - ba, ac - ca, bc - cb).$$

In this paper we construct, for a given term ordering $<$ on $K[x, y, z]$, an uncountable family of monomial orderings on A, which lift $<$, and the resulting Gröbner basis theory of I.

This work was inspired by [1]. Given a term ordering on the commutative polynomial ring $K[x_1, \ldots, x_n]$ and an ideal I, the authors lift the term ordering to an ordering on the free associative algebra $K\langle a_1, \ldots, a_n\rangle$. Two monomials in the free algebra are compared by first comparing their projections to the polynomial ring. If those are equal, then the words are compared lexicographically. The authors also lift the ideal I to an ideal in the free algebra by adding the commutator relations. A main result in [1] is that, after a generic change of coordinates in $K[x_1, \ldots, x_n]$, the lifted ideal in the free algebra has a finite Gröbner basis. The idea of studying ideals in commutative polynomial rings by considering a non-commutative presentation arises from work on the homology of coordinate rings of Grassmanians and toric varieties. See also [6]. One can now ask what happens if one chooses more exotic liftings of the commutative term ordering.

In the present paper, we construct other types of liftings. For each commutative term ordering on $K[x, y, x]$, we give uncountably many liftings to $K\langle a, b, c\rangle$. We then consider the commutator ideal, that is, the lifting of the zero ideal to the free algebra, and study its Gröbner bases with respect to the lifted monomial orderings. It is our hope that these monomial orderings will prove useful in applications to commutative problems.

Two central features of Gröbner basis theory, both commutative and non-commutative, are that it provides a set of normal forms and allows the solution of the ideal membership problem. In order to obtain both these features for a given ideal and a given set of generators it is not required that one start with a term ordering, that is, a total well-founded ordering on the set of monomials which is compatible with multiplication. If the Buchberger or Mora algorithm is performed with an ordering that is well-founded, and a set of generators for the ideal is created by this algorithm for which the reduction process modulo this set always terminates after finitely many steps in a normal form, then this generating set will solve the ideal membership problem and give a set of normal forms. It is not even necessary that the ordering be total. We believe that it might be very fruitful to study such "weak term orderings" for free algebras. In the commutative case, one does not actually obtain anything new, since each weak term ordering can be replaced by an actual term ordering with the same normal forms [8]. This paper shows that the result in [8] does not generalize to the noncommutative case.

One of the fundamental results in commutative Gröbner basis theory is that every ideal in a polynomial ring has only finitely many initial ideals (see, e.g., [9, Thm. 1.2]). It is well known that this result is false in the non-commutative theory. A survey of counterexamples can be found in [2]. We show that the initial ideals of the commutator ideal in A with respect to our monomial orderings are all distinct, so that the commutator ideal has at least uncountably many initial ideals, answering an open question in [2]. Since the difference between the commutative and non-commutative cases is the commutator ideal, it is not surprising that the differences between the two theories should manifest themselves there.

Since the commutator ideal is a binomial ideal with each generator consisting of the difference of two monomials, the quotient ring is the monoid ring over K for the free commutative monoid on three generators. Thus the Mora algorithm for the commutator ideal and the Knuth-Bendix algorithm for the monoid will produce the same set of normal forms if they start with the same ordering. So our theorem also shows that for the free commutative monoid on three generators, there are uncountably many distinct minimal complete rewriting systems with respect to our orderings.

All of the possible term orderings on the set of words over two generators have been classified in [4], [5], and [7]. While we have developed many more orderings for words over three letters, we do not know if repeating our constructions would allow us to find all of the Gröbner bases for the commutator ideal. It would be of interest to attempt the classification of all term orderings for three letters to answer this question.

2 Gröbner Bases and Rewriting Systems

Since the results in this paper can be interpreted both in the framework of Gröbner basis theory and that of rewriting systems, we give here a brief summary of relevant definitions and their relationship.

Let K be a field, let Σ be a finite set, let Σ^* be the free monoid on Σ, and let $A = K\langle\Sigma\rangle = K[\Sigma^*]$ be the free associative algebra over K on Σ.

A *rewriting system* over Σ is a set $R \subseteq \Sigma^* \times \Sigma^*$ of *replacement rules*, where an element (or rule) $(u, v) \in R$ is also written $u \to v$. In general, if $u \to v$, then whenever the word u appears inside a larger word, we will replace it with the word v; that is, for any $x, y \in \Sigma^*$, we write $xuy \to xvy$ and say that the word xuy is *rewritten* (or *reduced*) to the word xvy. An element $x \in \Sigma^*$ is *irreducible* or in *normal form* if it cannot be rewritten. The ordered pair (Σ, R) is a rewriting system for a monoid M if

$$\langle\ \Sigma \mid u = v \text{ if } (u, v) \in R\ \rangle$$

is a presentation for M.

A rewriting system (Σ, R) is *terminating* if there is no infinite chain $x \to x_1 \to x_2 \to \cdots$ of rewritings; that is, if the partial ordering defined by $x \geq y$ whenever $x \to \cdots \to y$ is well-founded. R is *confluent* if whenever a word x can be rewritten in two different ways to words y_1 and y_2, then there are rewritings from y_1 and y_2 to a common word z. When R is terminating, confluence is equivalent to saying that there is exactly one irreducible word representing each element of the monoid presented by the rewriting system. The system R is *complete* if it is both terminating and confluent. Finally, it is *minimal* if each right hand side, and each proper subword of a left hand side, of a rule is irreducible.

In this paper we will use the expression *monomial ordering* to denote a partial well-founded ordering on a set of monomials. A *term ordering* is a monomial ordering with the additional properties that it is total and compatible with multiplication; that is, whenever $x < y$ in the ordering, then $wxz < wyz$ also.

Let $<$ be a monomial ordering on $K\langle\Sigma\rangle$. If $f \in K\langle\Sigma\rangle$, the largest monomial of f with respect to $<$ is its *leading term*. A *Gröbner basis* for an ideal I of A is a subset G of I such that the ideal generated by the leading terms of elements in G is equal to the ideal generated by the leading terms of all elements of I. A Gröbner basis G is *reduced*, if no term of any polynomial in G is divisible by the leading term of a polynomial in G. In the following result we state the relationship between rewriting systems and Gröbner bases. Its proof is straightforward.

Theorem 2.1 Suppose M is the free commutative monoid generated by $\Sigma = \{a, b, c\}$. Let R be a minimal complete rewriting system for M, and let

$$G = \{u - v \mid u \to v \in R\}.$$

Then G is a reduced Gröbner basis for the commutator ideal I of A.

The Gröbner bases of I corresponding to distinct rewriting systems of M are also distinct. For more details on rewriting systems, Gröbner bases, and the connections between them for monoid rings, see [3].

3 Monomial Orderings and Initial Ideals

Let

$$\gamma : A = K\langle a, b, c\rangle \longrightarrow K[x, y, z]$$

be the canonical projection with kernel the commutator ideal I. In this section we will prove the following theorem.

Theorem 3.1 Let $<$ be a term ordering on $K[x,y,z]$. There exist monomial orderings $\prec_r$ on $A = K\langle a,b,c\rangle$, for each r in the interval $(0,\frac{1}{2}) \subset \mathbf{R}$, which are liftings of $<$, in the sense that, if $\gamma(w) < \gamma(w')$, then $w \prec_r w'$. Furthermore, the initial ideals of the commutator ideal I with respect to the orderings $\prec_r$ are all distinct. Hence, I has uncountably many reduced Gröbner bases. The reduced Gröbner bases for I with respect to $\prec_r$ are all infinite.

As noted in the previous section, each minimal complete rewriting system for the free commutative monoid M corresponds to a reduced Gröbner basis for the commutator ideal I. Thus the proof of Theorem 3.1 follows directly from the proof of the following theorem.

Theorem 3.2 Let $<$ be a term ordering on $K[x,y,z]$. There exist well-founded orderings $\prec_r$ on Σ^*, for each r in the interval $(0,\frac{1}{2}) \subset \mathbf{R}$, which are liftings of $<$, in the sense that, if $\gamma(w) < \gamma(w')$, then $w \prec_r w'$. Furthermore, the minimal complete rewriting systems for the free commutative monoid M on three generators corresponding to these orderings are all distinct and are all infinite. Hence, M has at least uncountably many minimal complete rewriting systems.

Let

$$r = .i_1 i_2 \ldots$$

be the binary expansion of a real number r in the interval $(0,\frac{1}{2})$, with $i_n \in \{0,1\}$ for all n, and $i_1 = 0$. (If r does not have a unique binary expansion, we simply chose such an expansion.) Let $m_1, m_2, \ldots$ be the indices for which $i_{m_j} = 1$, with $1 < m_1 < m_2 < \cdots$. Set $m_0 = 1$. Now associate the following rewriting system $R_r$to r:

$$\begin{array}{rcllr}
ab & \rightarrow & ba, & & (1)\\
bc & \rightarrow & cb, & & (2)\\
ca & \rightarrow & ac, & & (3)\\
ac^k b^j & \rightarrow & c^k b^j a & \text{whenever} \quad 1 \leq j \quad \text{and} \quad m_{j-1} \leq k < m_j, & (4^j)\\
c^{m_j} b^j a & \rightarrow & ac^{m_j} b^j & \text{whenever} \quad 1 \leq j, & (5^j)\\
ba^i c & \rightarrow & cba^i & \text{whenever} \quad 1 \leq i & (6^i)
\end{array}$$

Thus, R_r is an infinite rewriting system for M.

First consider the case that there are only finitely many such indices m_j, with m_J being the largest one. Set $m_j = \infty$, if $j > J$. In this case we have rules (4^j) for $1 \leq j \leq J+1$ and rules (5^j) for $1 \leq j \leq J$. Note that in this case, the rewriting system has the additional property that it is regular, in the sense that the set of word pairs given by the left and right hand sides of rules is a regular language accepted by a finite state automaton. The normal forms associated to the rewriting system are therefore also a regular language.

To each word w in Σ^* we associate an integer vector $\Psi(w)$ as follows. Each of the entries of this vector $\Psi(w)$ will be a "barrier ordering"; we will place barriers in w, and then use these barriers to associate a number to w.

To define the first entry $\alpha(w)$ of $\Psi(w)$, insert barriers in w to immediate the right of every letter c that occurs. For instance, if $w = abc^2baca$, then we place barriers as follows:

$$w = abc|c|bac|a.$$

Let $l_b(i)$ denote the number of occurrences of b to the left of the i-th barrier in w. Suppose there are n barriers in w. Now define

$$\alpha(w) = l_b(1) + \cdots + l_b(n).$$

For the example $w = abc^2baca$ above, we have

$$\alpha(w) = 1 + 1 + 2 = 4.$$

The other entries of $\Psi(w)$ are defined similarly, but the barriers will be placed after certain types of subwords rather than individual letters. First we introduce some notation. Given any word t in Σ^*, let $\tilde{t}$ be the word obtained from t by deleting all occurrences of a in t, and let $l_a(t)$ denote the number of occurrences of a in t. Suppose s is a word involving just the letters b and c, and w is a word in Σ^* as above. We can write

$$w = v_1 s_1 v_2 s_2 ... s_k v_{k+1}$$

where:

(i) $\tilde{s}_i = s$,

(ii) if t_i is a proper subword of s_i then $\tilde{t}_i \neq s$, and

(iii) k is maximal.

As in the definition of α, insert barriers into w immediately to the right side of each subword s_i; then count the number of occurrences of a to the left of each barrier; that is, define

$$L(s, w) = \sum_{i=1}^{k} l_a(v_1 s_1 ... v_i s_i).$$

Similarly, define

$$R(s, w) = \sum_{i=1}^{k} l_a(s_i v_{i+1} ... s_k v_{k+1}),$$

placing barriers to the left of each subword s_i and counting the number of a's to the right of the barriers.

Finally, define the vector $\Psi(w)$ by

$$\Psi(w) = (\alpha(w), L(c^{m_J} b^{J+1}, w), R(c^{m_J} b^J, w), ..., L(c^{m_1} b^2, w), R(c^{m_1} b, w), L(cb, w)).$$

We are now ready to define the ordering $\prec_r$ on Σ^*, using the commutative ordering $<$ via the projection

$$\gamma : K\langle a, b, c\rangle = K[\Sigma^*] \longrightarrow K[x, y, z],$$

and the lexicographic ordering on the integer vector $\Psi(w)$ associated to a word w. We assume, without loss of generality (by relabeling the letters a, b, c), that $y < x < z$ in $K[x, y, z]$, so that $b \prec_r a \prec_r c$.

Now let w, w' be two words.

Definition 3.3 Let $w \prec_r w'$ if

1. $\gamma(w) < \gamma(w')$, or
2. $\gamma(w) = \gamma(w')$ and $\Psi(w) <_{\text{lex}} \Psi(w')$, or
3. $\gamma(w) = \gamma(w'), \Psi(w) = \Psi(w')$ and $w <_{\text{lex}} w'$.

In this way we obtain a partial ordering on the words in Σ^*. We show in Proposition 3.5 that this ordering is in fact a total ordering.

Example 3.4 Let $r = .0101$, and let $w = cacb^2cac^2acbab^2c$. Then $J = 2$, $m_0 = 1$, $m_1 = 2$, and $m_2 = 4$. We need to compute

$$\left(\alpha(w), L(c^4b^3, w), R(c^4b^2, w), L(c^2b^2, w), R(c^2b, w), L(cb, w)\right).$$

We may write w as

$$w = c|ac|b^2c|ac|c|ac|bab^2c|;$$

Then

$$\alpha(w) = 0 + 0 + 2 + 2 + 2 + 2 + 5 = 13.$$

To compute $L(c^4b^3, w)$, write $v_1 = cacb^2$, $s_1 = cac^2acbab^2$, and $v_2 = c$. Then $w = v_1s_1v_2$, $\tilde{s}_1 = c^4b^3$, and conditions (i)–(iii) are met. In this case,

$$L(c^4b^3, w) = l_a(v_1s_1) = 4.$$

Computing $R(c^4b^2, w)$ next, write $v_1 = cacb^2$, $s_1 = cac^2acbab$, and $v_2 = bc$. Again $w = v_1s_1v_2$, but this time $\tilde{s}_1 = c^4b^2$, and conditions (i)-(iii) in the definition of $R(c^4b^2, w)$ are met. In this case,

$$R(c^4b^2, w) = l_a(s_1v_2) = 3.$$

The decomposition of w to compute $L(c^2b^2, w)$ is given by $w = v_1s_1v_2s_2v_3$ where $v_1 = 1$ (the empty word), $s_1 = cacb^2$, $v_2 = cac$, $s_2 = cacbab$, and $v_3 = bc$. Thus

$$L(c^2b^2, w) = l_a(v_1s_1) + l_a(v_1s_1v_2s_2) = 1 + 4 = 5.$$

Similar computations give $R(c^2b, w) = 6$ and $L(cb, w) = 4$. Assembling these numbers, we obtain

$$\Psi(w) = (13, 4, 3, 5, 6, 4).$$

Proposition 3.5. The partial ordering $\prec_r$ is a well-founded total ordering. Furthermore, the set R_r is a minimal complete rewriting system for the free commutative monoid M with respect to $\prec_r$.

Proof. We first show that the ordering is well-founded. The ordering $<$ on commutative polynomials is a term ordering, so it is well-founded. In Definition 3.3, $\Psi(w)$ and $\Psi(w')$ have the same length, so we can replace the ordering $<_{\text{lex}}$ by the well-founded ordering $<_{\text{lengthlex}}$ in part (2) of the definition without changing the ordering. Similarly, in item (3) two words w and w' will be compared using $<_{\text{lex}}$ only if $\gamma(w) = \gamma(w')$, so the words w and w' will have the same length. Then we can also replace the ordering $<_{\text{lex}}$ by the well-founded ordering $<_{\text{lengthlex}}$ in part (3) without altering the ordering $\prec_r$. Therefore $\prec_r$ is also a well-founded ordering.

To show that $\prec_r$ is a total ordering, let w and w' be words in Σ^*. We need to show that they are comparable. Since the lengthlex ordering is a total ordering, and item (3) of Definition 3.3 compares w and w' with this ordering, these two words must be comparable with the ordering $\prec_r$, also.

Next we show that the process of reduction modulo R_r will always terminate after finitely many steps. To do this, it is sufficient to show that if we rewrite a word w, using one of the rules of types (1)–(6) above, then the resulting word w' is such that $w' \prec_r w$.

Suppose that a word w is rewritten to a word w' using one of the rules. Observe first that applying any of the rules does not change the value of γ, so $\gamma(w) = \gamma(w')$. If one of the rules (2) or $(6^i), i \geq 1$, is applied, then $\alpha(w') < \alpha(w)$, so $w' \prec_r w$.

Now consider the rule (4^{J+1}), where m_J is the largest place for which a 1 occurs in the decimal expansion of r. Rewriting w to w' by applying rule (4^{J+1}) leaves α unchanged, so $\alpha(w') = \alpha(w)$. However, $L(c^{m_J} b^{J+1}, w') < L(c^{m_J} b^{J+1}, w)$, so this rule decreases Ψ and $w' \prec_r w$.

Applying rule (5^J) does not alter the value of α. Also, since rule (5^J) cannot move an a past any of the barriers used to compute $L(c^{m_J} b^{J+1}, \cdot)$, it will not alter its value. However, this rule does reduce $R(c^{m_J} b^j, \cdot)$, and hence decreases Ψ, so again $w' \prec_r w$.

Continuing inductively, we see that rewriting w to w' by rules $(4^j), j \leq J+1$ leaves $\alpha(w) = \alpha(w')$, $L(c^{m_J} b^{J+1}, w) = L(c^{m_J} b^{J+1}, w')$, $R(c^{m_J} b^J, w) = R(c^{m_J} b^J, w')$,..., $R(c^{m_j} b^j, w) = R(c^{m_j} b^j, w')$, and $L(c^{m_j} b^{j+1}, w) > L(c^{m_j} b^{j+1}, w')$. Similarly, applying a rule of the form $(5^j), j \leq J$ leaves $\alpha(w) = \alpha(w')$, $L(c^{m_J} b^{J+1}, w) = L(c^{m_J} b^{J+1}, w')$, $R(c^{m_J} b^J, w) = R(c^{m_J} b^J, w')$,..., $L(c^{m_{j+1}} b^{j+2}, w) = L(c^{m_{j+1}} b^{j+2}, w')$, and $R(c^{m_j} b^j, w) > R(c^{m_j} b^j, w')$. In each case, then, the rules $(4^j), (5^j)$ decrease Ψ, and if one of these rules is applied to rewrite w to w', then $w' \prec_r w$.

Finally, if rule (1) or (3) is applied to w, then $\alpha(w') = \alpha(w)$. For each index j, applying rule (1) or (3) either does not move past a barrier used to compute $L(c^{m_j} b^{j+1}, \cdot)$ or $R(c^{m_j} b^j, \cdot)$, or else it moves an a past a barrier in a way that will decrease the corresponding variable. Therefore $\Psi(w') \leq \Psi(w)$. However, rewriting by (1) and (3) decreases w lexicographically, since $b \prec_r a \prec_r c$, so $w' \prec_r w$. Thus, we have shown that the reduction process always decreases the well-founded ordering $\prec_r$ no matter what rule is applied, so this process will always terminate after finitely many steps.

Next we show that the reduction process results in normal forms. This is straightforward to verify by showing that R_r is confluent. To obtain the set of normal forms explicitly, let w be a word, which we write as

$$w = w_1(a,c)bw_2(a,c)b\cdots bw_n(a,c),$$

where the subwords $w_i(a,c)$ do not contain b. Using rule (3), we can rewrite each w_m in the form a^ic^k. Now, using rules (1)-(3) and (6^i), it is straightforward to check that we can rewrite w to $w' = a^ic^kb^ja^{i'}$. To rewrite w' further, we consider three cases.

First suppose that $j = 0$; then applying rule (3) repeatedly rewrites w' to the form a^ic^k. Next suppose that $j \geq 1$ and $k < m_j$. Let $1 \leq j' \leq j$ be an integer such that $m_{j'-1} \leq k < m_{j'}$. Then

$$w' = (a^ic^kb^{j'})b^{j-j'}a^{i'},$$

and we can rewrite w' to $c^kb^ja^{i+i'}$ using rules ($4^{j'}$) and (1). Finally, if $j \geq 1$ and $k \geq m_j$, then we write

$$w' = a^ic^{k-m_j}(c^{m_j}b^ja^{i'}),$$

which we can rewrite to $a^{i+i'}c^kb^j$, using rules (5^j) and (3). Thus, the set of normal forms is

$$\{a^ic^k | i,k \geq 0\} \cup \{c^kb^ja^i | i,k \geq 0,\ j \geq 1, \text{and} k < m_j\} \cup \{a^ic^kb^j | i,k \geq 0,\ j \geq 1, \text{and} k \geq m_j\}.$$

This completes the proof of the proposition.

Remark. What keeps $\prec_r$ from being a term ordering in the usual sense is that it is not compatible with multiplication. As an example, let $r = .001$, so that $J = 1, m_0 = 1, m_1 = 3$. Let $u = c^2$, $w = cab^2a$ and $w' = cba^2b$. Then $\gamma(w) = \gamma(w')$, and

$$\Psi(w) = (0,0,0,1) > (0,0,0,0) = \Psi(w'),$$

so that $w' \prec_1 w$. But $\gamma(uw) = \gamma(uw')$, and

$$\Psi(uw) = (0,1,2,1) < (0,2,2,0) = \Psi(uw').$$

In the proof of Theorem 3.2, it remains to consider the case of infinitely many 1's in the binary expansion of r. To define the ordering $\prec_r$ in this case, suppose that w and w' are any words, and let l be the length of the longest of the two words. Let r' be the real number whose first l binary digits are the same as those of r, and whose remaining digits are all zero. (Replace the $(l+1)$st digit with a 1, if necessary, to be sure that $0 < r' < .1 = \frac{1}{2}$.) Then define $w' \prec_r w$ if $w' \prec_{r'} w$. This automatically gives the property that $w' \prec_r w$ whenever $\gamma(w') < \gamma(w)$, since this is true for $\prec_{r'}$. Note that if $0 < r' < r'' < .1$ and both have finite binary expansions whose first l digits agree, then $w' \prec_{r'} w$ if and only if $w' \prec_{r''} w$; in other words, our orderings $\prec_r$ behave well under formation of limits of the numbers r. This implies that any infinite chain $w_1 \succ_r w_2 \succ_r \cdots$ corresponds to an infinite chain $w_1 \succ_{r'} w_2 \succ_{r'} \cdots$ for a real number $0 < r' < .1$ with a finite binary expansion. Well-foundedness of the ordering $\prec_r$ then follows from the well-foundedness of the orderings $\prec_{r'}$ in Proposition 3.5.

For any given word w of length l, the rules of R_r which can be applied in the process of reducing w to its normal form must also appear in $R_{r'}$; application of these rules strictly decreases the ordering $\prec_r$. Thus we can truncate the binary expansion of r after a finite number of 1's, and we can proceed as in the case of a finite binary expansion treated above; the reduction process on w must terminate after finitely many steps.

Finally, to show that the reduction process results in normal forms in the case when the expansion for r contains infinitely many 1's, we can again verify this by checking that R_r is confluent.

In summary, we have established a one-to-one correspondence between all real numbers in the interval $(0, \frac{1}{2})$ and monomial orderings, or weak term orderings, on the free monoid Σ^*. With respect to the monomial ordering $\prec_r$, the free commutative monoid M has the minimal complete rewriting system R_r defined above. Since $R_r \neq R_{r'}$ if $r \neq r'$, we have established a one-to-one correspondence between the set of real numbers in $(0, \frac{1}{2})$ and a set of complete rewriting systems of M. This completes the proof of Theorem 3.2 and therefore also that of Theorem 3.1.

References

[1] D. Eisenbud, I. Peeva, and B. Sturmfels, Non-commutative Gröbner bases for commutative algebras , *Proc. Amer. Math. Soc.* **126** (1998) 687–691.

[2] E. Green, T. Mora and V. Ufnarovski, The non-commutative Gröbner freaks, in *Symbolic Rewriting Techniques* (Ascona, 1995), 93–104, Progr. Comput. Sci. Appl. Logic **15**, Birkhäuser, Basel, 1998.

[3] K. Madelener and B. Reinert, String rewriting and Gröbner bases – a general approach to monoid and group rings, *Symbolic Rewriting Techniques* (Ascona, 1995), 127–180, Progr. Comput. Sci. Appl. Logic **15**, Birkhäuser, Basel, 1998.

[4] U. Martin and E. Scott, The order types of termination orderings on monadic terms, strings and multisets, *Eighth Annual IEEE Symposium on Logic in Computer Science* (Montreal, PQ, 1993), 356-363, IEEE Comput. Soc. Press, Los Alamitos, CA, 1993.

[5] Martin and E. Scott, The order types of termination orderings on monadic terms, strings and multisets, *J. Symbolic Logic* **62** (1997) 624–635.

[6] I. Peeva, V. Reiner and B. Sturmfels, How to shell a monoid, *Math. Ann.* **310** (1998) 379–393.

[7] S. Perlo-Freeman and P. Pröhle, Scott's conjecture is true, position sensitive weights, Rewriting Techniques and Applications (Sitges, 1997), 217–227, Lecture Notes in Comput. Sci., 1232, Springer, Berlin, 1997.

[8] A. Reeves and B. Sturmfels, A note on polynomial reduction, *J. Symbolic Comput.* **16** (1993) 273–277.

[9] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, University Lecture Series **8**, American Mathematical Society, Providence, R. I., 1996.