Isoperimetric inequalities for soluble groups

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Abstract: We approach the question of which soluble groups are automatic. We describe a class of nilpotent-by-abelian groups which need to be studied in order to answer this question. We show that the nilpotent-by-cyclic groups in this class have exponential isoperimetric inequality and so cannot be automatic.

Key words: Isoperimetric function, soluble groups.

AMS (MOS) Subj. Class.: 20F65, 20F16, 20F69

1. Introduction

Automatic structures for finitely presented groups are algorithmic and geometric properties which provide solutions for the word problem. Isoperimetric functions for groups measure how efficiently the word problem can be solved; for an automatic group, the isoperimetric function grows at most quadratically as a function of word length, so that the word problem can be solved in at most quadratic time ([10], Theorem 2.3.12).

The nilpotent groups which are automatic are known; a theorem of Holt states that a nilpotent group is automatic if and only if it is virtually abelian ([10], Theorem 8.2.8). In this paper we investigate the question of whether this theorem still holds if nilpotent groups are replaced with the larger class of soluble groups.

All automatic groups satisfy the homological finiteness condition FP_∞ ([1], Corollary 1), so it suffices to consider only the soluble groups which are of this type. A deep theorem of Kropholler ([14], Corollary to Theorem B) implies that a soluble group of type FP_∞ must be constructible; this means that the group can be built up from the

† The second author wishes to thank the National Science Foundation for financial support from grants INT-9223826 and DMS-9623088.

trivial group using finite extensions and HNN extensions. So it suffices to consider constructible groups if we wish to investigate automatic soluble groups.

Section 2 of this paper includes basic definitions of isoperimetric functions and constructible groups. In Section 3 we show that every constructible soluble group can be obtained by the following operations, in the given order: Start with a finitely generated nilpotent group, form one HNN extension in which the base group coincides with one of the associated subgroups (that is, an *ascending* HNN extension), form a finite number of split extensions by infinite cyclic groups, and, finally, form a finite extension.

The results of Section 3 show that every possible candidate for an automatic soluble group must be obtained using this sequence of operations. If G is a finite index subgroup of a group H, then G is automatic if and only if H is automatic([10], Theorem 4.1.4), so we can neglect the final operation.

In the rest of the paper, we concentrate on the special case in which the group G is a single ascending HNN extension of a finitely generated torsion-free nilpotent group. In Section 4 we determine a matrix criterion for when the group G is virtually nilpotent or polycyclic, and note that G is polycyclic if and only if the HNN extension is actually a split extension. In Sections 5 and 6 we investigate lower bounds for isoperimetric functions for G, and obtain the following theorem.

Main Theorem. *Suppose that G is an ascending HNN extension of a finitely generated torsion-free nilpotent group. Then G is automatic if and only if G is virtually abelian. Moreover, if G is not virtually nilpotent, then the isoperimetric function for G is at least exponential, and if G is not polycyclic, the abelianized isoperimetric function for G is also at least exponential.*

Since an automatic group has isoperimetric function which grows at most quadratically as a function of word length, the second sentence of the Main Theorem follows directly from Holt's theorem and the isoperimetric function results in the third sentence.

In [8], Bridson and Gersten have proved that these results are true in the special case that G is a split extension of a free abelian group of finite rank by an infinite cyclic group. We use topological techniques similar to those in the proof from [8] in Sections 5 and 6.1 to prove the lower bound on the isoperimetric function for the groups in the Main Theorem.

In Section 6.2, we prove the lower bound on the abelianized isoperimetric function in the case that G is not polycyclic. The connection between the abelianized and "usual" isoperimetric functions is an open problem; while it is known that the isoperimetric function is at least as large as the abelianized isoperimetric function for any group [4], no example has been found of a group for which these two functions are not equivalent. If the abelianized isoperimetric function has strictly smaller growth than the isoperimetric function for the groups we consider, then the lower bound on the abelianized isoperimetric function in the Main Theorem gives a stronger result than that in Section 6.1. In either case, the proof in Section 6.2 gives a second, substantially different, proof of the Main Theorem when G is not polycyclic, which includes the development of algebraic methods to find lower bounds on the abelianized isoperimetric function. For these reasons, we include the details of this proof.

The Main Theorem determines which soluble groups are automatic only in the special case in which the group is a single ascending HNN-extension of a torsion-free nilpotent group. A generic example of a constructible soluble group, presented in Section 3, for which our methods applied routinely do not work, is the group

$$\langle s, t, a \mid [s,t] = 1, a^s = a^2, a^t = a^3 \rangle.$$

We are left with the following conjecture.

Conjecture. *A soluble group is automatic if and only if it is virtually abelian.*

2. Background and notation

2.1. Isoperimetric functions

Suppose that G is a group with a finite presentation $G = \langle A \mid Y \rangle$. Let w be an element in the free group $F = F(A)$ on the set A. Then under the canonical map $F \to G$, the element w maps to an element of G, which we will also denote w. To make it clear when we are considering w as an element of F or G, we will use subscripts on equations; for example, the equation $w =_G 1$ means that w represents the trivial element of G.

For an element $w \in F$, the equality $w =_G 1$ is true if and only if

$$w =_F \prod_{k=1}^{q} v_k r_k^{\epsilon_k} v_k^{-1} =_F \prod_{k=1}^{q} (r_k^{\epsilon_k})^{v_k},$$

for some $\epsilon_k \in \{\pm 1\}$, $v_k \in F$ and relators $r_k := a_k b_k^{-1}$ where $a_k = b_k$ is a relation in Y. In this case the *area* of the element w is defined to be the minimum value of q over all such representations of w. The *length* of w is the length of a reduced word which represents w over the alphabet $A \cup A^{-1}$.

The expression for w as a product of conjugates of relators has an associated *van Kampen diagram*; this is a planar 2-dimensional combinatorial CW-complex with directed edge labels given by elements of A; the boundary edges spell out the word w, and the boundary of each 2-cell is labeled by a relator formed from a relation in Y. See [8] for definitions and details. The area of this diagram is the number of 2-cells, and the area of w is then the area of the diagram for w with the fewest 2-cells.

An isoperimetric function measures the relation between the area and the length of words that represent the trivial element of the group. The *isoperimetric function* or *Dehn function of the presentation* $\mathcal{P} = \langle A \mid Y \rangle$ of the group G is the function $f : \mathbf{N} \to \mathbf{N}$ defined by

$$f(n) = \max\{area(w) \mid w =_G 1, \, length(w) \leq n\}.$$

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two different presentations for the group G, then the corresponding isoperimetric functions f_1 and f_2 are equivalent, in the sense that there is a constant C such that $f_1(n) \leq Cf_2(Cn+C)+Cn+C$ and $f_2(n) \leq Cf_1(Cn+C)+Cn+C$ for every $n \in \mathbf{N}$ [2], [7], [11]. The *isoperimetric function* or *Dehn function for the group* G, then, refers to the equivalence class of any isoperimetric function for a presentation of G.

In [4], an *abelianized isoperimetric function* has been defined, by considering the corresponding element $w[R,R]$ in the relation module $R/[R,R]$. The area is then calculated in the same way as above, using a minimal number of factors in an expression for $w[R,R]$ as a product of (cosets of) conjugates of the defining relators. This abelianized isoperimetric function, like the Dehn function, is independent of the presentation chosen, up to the same equivalence. In order to find lower bounds for this function, the Fox free differential calculus can be applied (see [9], p. 45-46 for the definition and details). To do this, the free derivatives of expressions for w as a word in the generators A of the group and as a product of conjugates of relators are computed and compared. We refer the reader to [4] for details.

2.2. Nilpotent groups

Suppose that N is a group. The *lower central series* of N is defined recursively by $\gamma_1(N) = N$, and for $i \geq 1$, $\gamma_{i+1}(N) = [\gamma_i(N), N]$, the subgroup generated by $\{aba^{-1}b^{-1} \mid a \in \gamma_i(N), \, b \in N\}$. For ease of notation later in the paper, the commutator subgroup $\gamma_2(N)$ is also denoted N', and the groups $\gamma_i(N)$ are also denoted N_i. The group N is a *nilpotent group of class c* if $\gamma_{c+1}(N) = 1$ and $\gamma_c(N) \neq 1$.

The *isolator* $I_N(L)$ of a subgroup L of a nilpotent group N is

$$I_N(L) = I(L) = \{x \in N \mid x^m \in L \text{ for some } m \neq 0\}.$$

In nilpotent groups, the isolator of a subgroup is a subgroup. We shall also say that a subgroup equal to its own isolator is *isolated.* It is easily verified that an isolator subgroup is isolated. For more information, see Chapter 4 of [12].

2.3. Constructible groups

A group G is said to be 0-*constructible* if it is finite. It is said to be n-*constructible* if it has a subgroup of finite index which is the fundamental group of a finite graph of groups of which the edge and vertex groups are $(n-1)$-constructible. The group is *constructible* if it is n-constructible for some finite n. In particular, constructible groups are finitely presented and of type FP_∞.

We shall be interested only in soluble constructible groups where the description can be made much simpler. Here we need to use only finite extensions and HNN extensions where one of the associated subgroups is equal to the base group. In fact, a major consequence of Proposition 3.1 will be to give an even simpler description of the necessary constructions.

We refer to [3] for details and further information about both constructible and soluble constructible groups.

3. A description of constructible soluble groups

Our first result is a general description of constructible soluble groups.

Proposition 3.1. *If H is a constructible soluble group, then there is a normal series*

$$\{1\} \le M \le K \le G \le H$$

such that G has finite index in H, G/M is torsion-free abelian, M is torsion-free nilpotent, K/M is infinite cyclic and K is constructible.

Proof. Observe that H is finitely presented, of finite rank, and torsion-free-by-finite (see, for example, the remark following Lemma 3 of [3]). Thus H is nilpotent-by-abelian-by-finite (see the proof of Theorem 10.38 of [16]) and so we can choose normal subgroups M and G of H so that M is torsion-free nilpotent, G/M is torsion-free abelian and H/G is finite. Since H is finitely generated, both G and G/M must also be finitely generated.

All that remains is to find the subgroup K with the desired properties. In order to do this, we consider the Bieri-Strebel invariant $\Sigma^c_{M_{\mathrm{ab}}}$ where M_{ab} is the quotient of M by its commutator subgroup, considered as a module for the finitely generated abelian quotient group $Q = G/M$ of G. We will use [6] as our principle reference for the Bieri-Strebel invariant. Denote by $E(Q)$ the set of all homomorphisms from Q to the additive group of the real numbers. Define two such homomorphisms ϕ_1 and ϕ_2 to be *equivalent* if $\phi_1 = \lambda\phi_2$ for some positive real number λ. This is an equivalence relation and we shall denote the equivalence class of $v \in E(Q)$ by $[v]$. Set $S(Q)$ to be the quotient of $E(Q) - \{0\}$ by this equivalence relation. Note that we can identify $E(Q)$ with $\mathbf{R}^n$ where n is the rank of the free abelian group Q; we can then identify $S(Q)$ with an $(n-1)$-sphere. The invariant $\Sigma^c_{M_{\mathrm{ab}}}$ is a subset of $S(Q)$. Note that the invariant $\sigma(H)$ used in [6] can be identified with both $\sigma(G)$ and $\Sigma^c_{M_{\mathrm{ab}}}$.

Theorem 5.2 of [6] says that a finitely generated soluble group L is constructible if and only if L is nilpotent-by-abelian-by-finite and $\sigma(L)$ is contained in an open hemisphere; moreover, if so, $\sigma(L)$ is finite and consists of a discrete set of points. Because H is a finitely generated soluble constructible group, this theorem applies, so $\Sigma^c_{M_{\text{ab}}}$ is a finite set of discrete points lying in an open hemisphere of $S(Q)$. Since $\Sigma^c_{M_{\text{ab}}}$ lies in an open hemisphere of $S(Q)$, there will be some $q \in Q$ such that $v(q) > 0$ for all $[v] \in \Sigma^c_{M_{\text{ab}}}$; it is straightforward to see that we can choose q so that $< q >$ is equal to its isolator in Q. Choose $t \in G$ so that the image of t in Q is q and denote by K the group generated by M and t.

Finally, we need to check that this subgroup K of G satisfies the desired properties. By definition of K, K/M is infinite cyclic, generated by t. In order to show K is constructible, we again employ Theorem 5.2 of [6]. By definition, K is a nilpotent-by-abelian subgroup of H.

In order to show that K is finitely generated, we consider the set $S(Q, < q >)$ of elements of $S(Q)$ which are zero on $< q >$. Observe that $S(Q, < q >) \cap \Sigma^c_{M_{\text{ab}}}$ is empty and so, by Corollary 4.5 of [5], M_{ab} is finitely generated as a $< q >$-module. Thus there is a finitely generated subgroup N of M so that $M = N^{<t>}M'$. By (for example) Corollary 1 to Lemma 2.6 of [12], $N^{<t>} = M$ and it follows that K is finitely generated by the finite set of generators of N together with t.

The embedding $< q > \to Q$ yields a surjection $\rho : S(Q, < q >)^c \to S(< q >)$ by restriction of homomorphisms. The fact that $v(q) > 0$ for all $[v] \in \Sigma^c_{M_{\text{ab}}}$ tells us that $\rho(\Sigma^c_{M_{\text{ab}}})$ consists of a single point. Proposition 1.2 of [6] then says that

$$\rho(\Sigma^c_{M_{\text{ab}}}) = \Sigma^c_{M_1}$$

where M_1 is M_{ab} regarded as a $< q >$-module. Therefore $\sigma(K) = \Sigma^c_{M_1}$ has one point.

This shows that $\sigma(K)$ must be contained in an open hemisphere, so it follows, again from Theorem 5.2 of [6], that K is constructible. □

Thus each constructible soluble group can be obtained from a finitely generated torsion-free nilpotent group by taking first an HNN extension to obtain K, then a finite number of split extensions by infinite cyclic groups and finally a finite extension. Observe that if such a group is automatic, so also is the subgroup of finite index described in the previous sentence.

4. A characterization of G in terms of matrices

In this section and the remainder of the paper, we restrict to the special case in which the group G described in Proposition 3.1 is a single ascending HNN extension of a torsion-free nilpotent group. Then we can present G as

$$G = \langle t, a_1, ..., a_l \mid a_i^t = x_i \ (1 \le i \le l),\ y_j = 1 \ (1 \le j \le p)\rangle,$$

where the x_i and y_j are words in $\{a_1^{\pm 1}, ..., a_l^{\pm 1}\}$ and the subgroup $N = \langle a_1, ..., a_l\rangle$ is a torsion-free nilpotent group of class c. Then $G = N*_\phi$, where $\phi : N \to N$ is the homomorphism given by $\phi(w) = twt^{-1}$ for each $w \in N$.

Lemma 4.1. *Let $(N_i)^q$ denote the subgroup of N_i generated by the set $\{x^q \mid x \in N_i\}$. For all $i \geq 1$, there exists a natural number k_i such that*

$$(N_i)^{k_i} \subseteq (\phi(N))_i \,.$$

Proof. Since ϕ is the restriction of a conjugation automorphism on G to the subgroup N, the map ϕ is a monomorphism on N. Then Lemma 10 of [3] says that the index $|N : \phi(N)|$ is finite. Theorem 4.4 of [12] can now be applied to show that the index $j_i = |N_i : \phi(N_i)|$ is also finite. Let $k_i = j_i!$, and suppose that $x \in N_i$. Then the cosets $\phi(N_i)$, $x\phi(N_i)$,..., $x^{j_i}\phi(N_i)$ cannot all be distinct, so $x^m\phi(N_i) = x^n\phi(N_i)$ for some $0 \leq m < n \leq j_i$, and therefore $x^d \in \phi(N_i)$ where $0 < d = n - m < j_i$. Then $x^{k_i} \in \phi(N_i)$ also, so $(N_i)^{k_i} \subseteq \phi(N_i)$. Since $\phi(N_i) = (\phi(N))_i$, this shows that $(N_i)^{k_i} \subseteq (\phi(N))_i$. □

Proposition 4.2. *For each index i, the map ϕ induces a homomorphism $\chi_i : N/N_i \to N/N_i$ and a monomorphism $\psi_i : N/I(N_i) \to N/I(N_i)$.*

Proof. That ϕ induces the homomorphisms χ_i and ψ_i follows from the easily verified facts that $\phi(N_i) \subseteq N_i$ and $\phi(I(N_i)) \subseteq I(N_i)$.

Now $uI(N_i) \in \ker\psi_i$ if and only if $\phi(u) \in I(N_i)$. This, in turn, happens if and only if $\phi(u^q) = \phi(u)^q \in N_i$ for some q. But then, by Lemma 4.1,

$$\phi(u^{qk_i}) \in (N_i)^{k_i} \subseteq (\phi(N))_i = \phi(N_i).$$

Hence, as ϕ is injective, $u^{qk_i} \in N_i$ and so $u \in I(N_i)$. Thus ψ_i is injective. □

Proposition 4.3. *G is polycyclic (respectively, nilpotent-by-finite) if and only if $G/I(M')$ is polycyclic (respectively, nilpotent-by-finite).*

Proof. One direction of the implications is clear. Recall that M is nilpotent, and note that $I(M')$ refers to $I_M(M')$.

Assume that $G/I(M')$ is nilpotent-by-finite; say G_1 is a normal subgroup of G having finite index so that $G_1/I(M')$ is nilpotent. By (for example) Theorem 2.5 of [12], the product of two nilpotent normal subgroups is again nilpotent and so we can assume that $G_1 \geq M$. But then G_1/M' is finite-by-nilpotent. Since it is also finitely generated and metabelian, it is residually finite (see Theorem 1 of [13]) and so also nilpotent-by-finite. Thus there is a normal subgroup G_2 of finite index in G so that G_2/M' is nilpotent. Again we can assume that $G_2 \geq M$. Now we can apply a result of Philip Hall (appearing as Lemma 3.7 bis of [12]) to show that G_2 is nilpotent and so G is nilpotent-by-finite.

Assume that $G/I(M')$ is polycyclic. Since M/M' is of finite rank, its torsion-subgroup is finite and so G/M' is polycyclic. Theorem 3 (special case (iii)) of [17] now tells us that G is polycyclic. □

Because N is finitely generated, $N/I(N')$ will be a finitely generated free abelian group, of some rank m. Thus the monomorphism $\psi_2 : N/I(N') \to N/I(N')$ can be represented by a square $m \times m$ matrix which we shall denote by A_ϕ. The next step is to transfer the properties of G into properties of this matrix.

Proposition 4.4. *(a) G is nilpotent-by-finite if and only if every eigenvalue of A_ϕ has absolute value 1.*

(b) G is polycyclic if and only if the determinant of A_ϕ is ± 1.

Proof. (a) The characteristic polynomial of A_ϕ is a monic polynomial with integer coefficients. Thus the roots of this polynomial, that is the eigenvalues of A_ϕ, are algebraic integers. Note that an algebraic integer is a root of unity if and only if all of its conjugates are of absolute value one (see, for example, 9.B of [15]). Thus all eigenvalues of A_ϕ are of absolute value 1 if and only if they are all roots of unity. This, in turn, happens if and only if, for some natural number k, all of the eigenvalues of A_ϕ^k are 1; that is, A_ϕ^k is unipotent. It is easy to check (see [18], Proposition 1.10 for details) that A_ϕ^k is unipotent for some k if and only if the induced action of t^k on each quotient M_i/M_{i+1} is trivial (that is, t^k *acts nilpotently on* M), and hence $G/I(M')$ is nilpotent-by-finite. We can now invoke Proposition 4.3 to complete the argument.

(b) Observe that the determinant of A_ϕ is ± 1 if and only if A_ϕ is invertible. This will happen if and only if ψ_2 is an automorphism. But $N/I(N')$ is a finitely generated abelian group and it is easily verified that an HNN-extension of the type we consider here is polycyclic if and only if it is a split extension; that is, if ψ_2 is an automorphism. Thus the determinant of A_ϕ is ± 1 if and only if $G/I(M')$ is polycyclic and we can again invoke Proposition 4.3 to complete the proof. □

Since $N/I(N')$ is a free abelian group of rank m, the generators $\{a_1, ..., a_l\}$ of N can be chosen so that $N/I(N')$ is the free abelian group with generators given by $\{a_1 I(N'), ..., a_m I(N')\}$ where $m \leq l$.

The following Proposition will be used in Sections 5 and 6.

Proposition 4.5. *Suppose that one of the eigenvalues of the $m \times m$ matrix A_ϕ has absolute value greater than 1. Then for some index $h \leq m$, there is an index $j \leq m$ such that the sum $\mu(n)$ of the exponents of the occurrences of a_j in any word over $\{a_1, ..., a_l\}$ representing $\phi^n(a_h)$ grows at least exponentially with n.*

Proof. Let λ be an eigenvalue of A_ϕ with $|\lambda| > 1$. Over the field of complex numbers, A_ϕ has an eigenvector v with eigenvalue λ which can be written as a linear combination (with coefficients in $\mathbf{C}$) of $\{a_1 I(N'), ..., a_m I(N')\}$. Now $A_\phi^n(v) = \lambda^n v$ shows that the absolute value of the coefficient of v grows exponentially with n. Therefore, for some h and j, the absolute value of the coefficient of $a_j I(N')$ in the expression for $A_\phi^n(a_h I(N'))$ must also grow at least exponentially. For this h and j the sum $\mu(n)$ of the exponents of the occurrences of a_j in any word over $\{a_1, ..., a_l\}$ representing $\phi^n(a_h)$ grows at least exponentially with n. □

5. Proof of the Main Theorem when G is polycyclic

As in the last section, suppose that N is a nilpotent group of class c, G is an ascending HNN extension $G = N*_\phi$ with stable letter t, and ψ_2 is the homomorphism induced by ϕ (defined in Section 4). In this section we will further assume that G is polycyclic and that G is not virtually nilpotent.

Proposition 3.1 implies that we may assume in this case that N is a normal subgroup of G. The homomorphism $N \to N$ given by $w \mapsto t^{-1}wt$ is then ϕ^{-1}, inducing the homomorphism ψ_2^{-1}. Proposition 4.4 says that the matrix A_ϕ representing the homomorphism ψ_2 must have an eigenvalue with absolute value greater than 1, and since ψ_2 is invertible, the inverse homomorphism ψ_2^{-1} also must have an eigenvalue with absolute value greater than 1.

Use Proposition 4.5 to choose elements b and d in N such that the lengths in N of $\phi^n(b)$ and $\phi^{-n}(d)$ both grow at least exponentially with n. Note that $\phi^n(b^{-1})$ and $\phi^{-n}(d^{-1})$ also both grow at least exponentially with n. Suppose that $\gamma(n)$ is the minimum of the lengths of these four elements in N.

Let $w(n) = w \in F$ be defined by

$$w(n) := [b^{t^{2n}}, d, d, ..., d] =_F [\cdots[b^{t^{2n}}, d], d]\cdots], d]$$

where the element d occurs c times. Since N is nilpotent of class c, $w =_G 1$. The element w can be written as

$$w =_F t^{2n}bt^{-2n}d^{\beta_1}t^{2n}b^{-1}t^{-2n}d^{\beta_2}\cdots t^{2n}bt^{-2n}d^{\beta_{2^c-1}}t^{2n}b^{-1}t^{-2n}d^{\beta_{2^c}},$$

where $\beta_i \in \{\pm 1\}$, $\beta_1 = 1$, and $\beta_{2^c} = -1$.

To analyze the area of w, we will extend the geometric methods of [8]. The van Kampen diagram for w can be viewed as a polygon with 2^c segments, each consisting of a sequence of edges labeled with $t^{2n}b^{\pm 1}t^{-2n}d^{\pm 1}$. Choosing one of the vertices of the 2^c-gon as a basepoint and following the segments $t^{2n}bt^{-2n}d^{\beta_1}$, then $t^{2n}b^{-1}t^{-2n}d^{\beta_2}$, etc. around in a counterclockwise direction, we will refer to the segments in order as the first, second, ... , and 2^cth segment.

In [8], Bridson and Gersten showed that in this van Kampen diagram the boundary edges labeled with t and t^{-1} must be connected to one another by (possibly empty) "t-corridors". A t-corridor consists of a pair of (possibly empty) words in the generators $\{a_1, ..., a_l\}$ for N labeling the two sides of a row of rectangular 2-cells, with edges connecting the two sides all labeled by t. These t-corridors may not cross one another, and each connects a t along the boundary to a t^{-1} elsewhere on the boundary, in a one-to-one fashion.

Consider the corridor corresponding to the first t in w. This t edge starts at the basepoint and points in a counterclockwise direction, so the corridor must go to one of

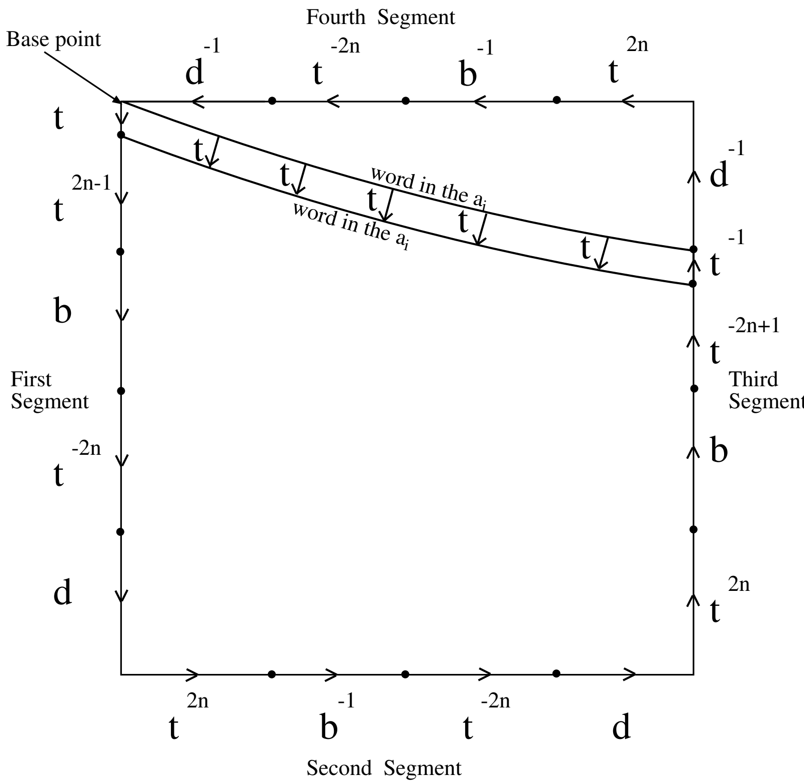

Figure 1. Van Kampen diagram for w when $c = 2$ and $k = 3$.

the copies of t^{-1} on the polygon. Suppose this corridor goes to the kth segment on the polygon. See Figure 1 in the case when $c = 2$ and $k = 3$.

Since t-corridors are not allowed to cross, this means that all of the corridors corresponding to the t edges in the second segment must go to segments between 1 and k, inclusive. Therefore, every corridor starting at a t in the second segment may travel at most $k - 1$ segments away from the second segment in either direction. Repeating this procedure inductively shows that there is a segment for which all of the corridors starting at a t in that segment go to either the same segment, or else the immediately prior segment. Suppose this segment is the ith segment. See the example in Figure 2 in which $c = 2$, $n = 2$, and $i = 3$.

The corridors corresponding to the $2n$ copies of t in the ith segment must all go to copies of t^{-1} in either the ith or $(i-1)$th segment. So at least n of them must go to the same segment. In the first case, if n corridors starting at t's in the ith segment go to copies of t^{-1} in the ith segment, then the word $t^n b^{\pm 1} t^{-n}$ along the diagram boundary is connected by n t-corridors, and the word v along the outermost opposite side of the n t-corridors is a word in the generators $\{a_1, ..., a_l\}$ for N representing $t^n b^{\pm 1} t^{-n}$; see Figure 3. Recall that b was chosen so that the word v must have length at least $\gamma(n)$. Each of the edges in v has a 2-cell attached in the t-corridor above it. If C is the length of the longest relator, then there are at least $\gamma(n)/C$ 2-cells attached to the edges of v,

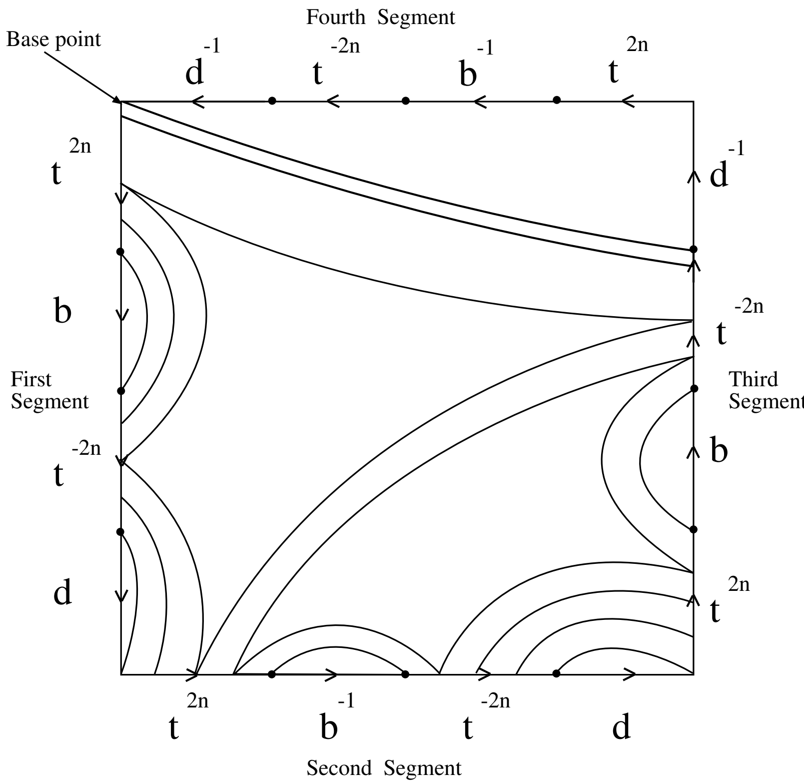

Figure 2. t-corridors and travel distances.

and so the area of this van Kampen diagram for w will be at least $\gamma(n)/C$. Similarly, in the second case, if n corridors starting at t's in the ith segment all go to copies of t^{-1} in the $(i-1)$th segment, then the word $t^{-n}d^{\pm 1}t^n$ along the boundary is connected by n t-corridors, and the word v along the opposite side of the n t-corridors is again a word in the generators for N with length at least $\gamma(n)$; see Figure 3. So in this case also, the area of the diagram is at least $\gamma(n)/C$.

Thus the area of $w = w(n)$ is at least $\gamma(n)/C$, where C is a constant and $\gamma(n)$ grows at least exponentially with n; so $area(w(n))$ grows at least exponentially with n. Also, the length of w is $(4n+2)2^c$, so $length(w(n))$ is a linear function of n. Therefore the isoperimetric function for G must also grow at least exponentially. This proves the Main Theorem in the case when G is polycyclic.

6. Proof of the Main Theorem when G is not polycyclic

As in the last two sections, suppose that G is an ascending HNN extension $G = N*_\phi$ with stable letter t, where N is a torsion-free nilpotent group of class c. In this section we will further assume that G is not polycyclic.

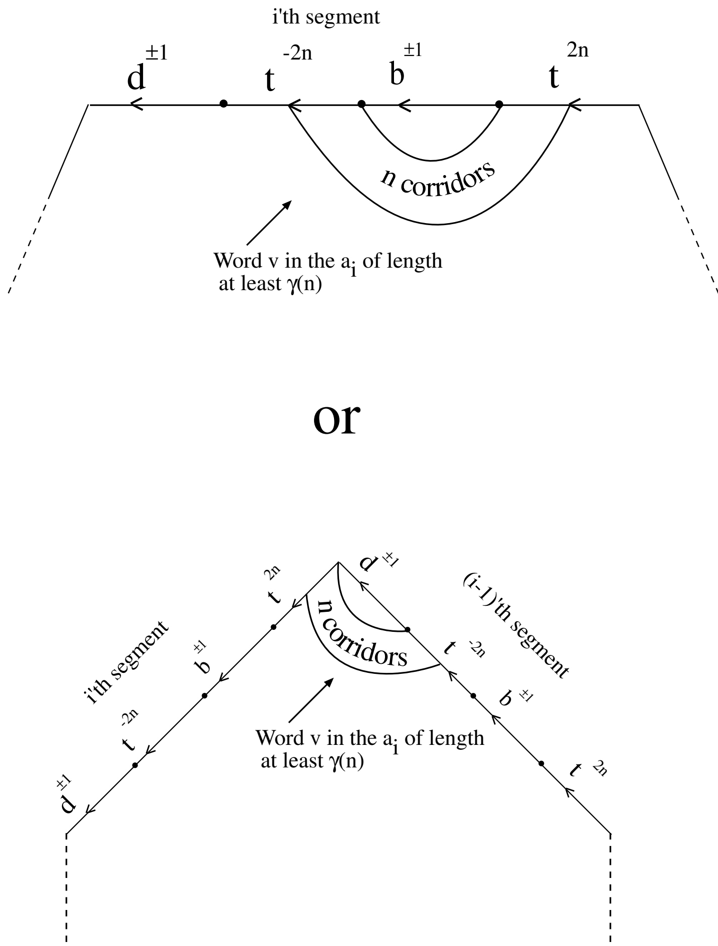

Figure 3. The t-corridors of the ith segment.

6.1. Isoperimetric functions

Most of the proof in Section 5 for the polycyclic case can be applied to prove that the isoperimetric function for the group G is at least exponential when G is not polycyclic, also. The main difference appears in the choice of the element d used to define the element $w(n)$.

Since G is not nilpotent-by-finite, Propositions 4.4 and 4.5 say that there is an element b in N such that the lengths in N of both $\phi^n(b)$ and $\phi^n(b^{-1})$ grow at least exponentially with n; let $\gamma(n)$ be the minimum of the lengths of these two elements in N. Since G is not polycyclic, Proposition 4.4 also says that the determinant of A_ϕ is not ± 1, so ϕ is not onto. Then there is an element d of N such that $d \neq \phi(d')$ for any element $d' \in N$, so $t^{-1}dt \notin N$. This implies that $t^{-n}d^{\pm 1}t^n \notin N$ for all $n \geq 1$.

For this new choice of elements b and d, define the element $w(n) = w \in F$ to be

$$w(n) := [b^{t^{2n}}, d, d, ..., d] =_F [\cdots [b^{t^{2n}}, d], d] \cdots], d]$$

as in the proof in Section 5. The rest of the proof in that section applies exactly as before, to show that any van Kampen diagram for w must have an ith segment for which the t-corridors corresponding to n copies of t in the ith segment either all go to copies of t^{-1} in the ith segment or all go to copies of t^{-1} in the $(i-1)$th segment (see Figure 3). If n corridors starting at a t in the ith segment go to copies of t^{-1} in the $(i-1)$th segment, then the word labeling the bottom side of the t-corridors (see the bottom section of Figure 3) must be a word in the generators a_i of N representing the element $t^{-n}d^{\pm 1}t^n$ of G. However, with the new choice of d, $t^{-n}d^{\pm 1}t^n \notin N$, so this cannot happen. Therefore n copies of t in the ith segment must go to copies of t^{-1} in the ith segment. As in the proof in Section 5, this implies that the area of $w(n)$ must be at least as large as the function $\gamma(n)/C$, which grows at least exponentially with n, but the length of $w(n)$ is linear, so the isoperimetric function for G must also grow at least exponentially if G is not polycyclic.

6.2. Abelianized isoperimetric functions

Let χ_i and ψ_i be the homomorphisms induced by ϕ (defined in Section 4). In the remainder of this section, we will denote the free abelian group $N/I(N')$ by $\tilde{N}$. The generators $\{a_1, ..., a_l\}$ of N can be chosen so that $\tilde{N} \cong \mathbf{Z}^m$ is the free abelian group with generators $\{a_1 I(N'), ..., a_m I(N')\}$ where $m \leq l$.

Since G is not polycyclic, Proposition 4.4 shows that the matrix A_ϕ must have an eigenvalue with absolute value greater than 1, and the map ψ_2 is not onto.

In this case $|\tilde{N} : \psi_2(\tilde{N})| > 1$. We can choose an integer k for which $|\tilde{N} : \psi_2^k(\tilde{N})| = |\tilde{N} : \psi_2(\tilde{N})|^k > 2^c$. The group

$$K = \langle s, a_1, ..., a_l | a_i^s = \phi^k(a_i) \ (1 \leq i \leq l), \ \ y_j(a_1, ..., a_l) = 1 \ (1 \leq j \leq p) \rangle$$

is a finite index subgroup of the group

$$G = \langle t, a_1, ..., a_l | a_i^t = \phi(a_i) \ (1 \leq i \leq l), \ \ y_j(a_1, ..., a_l) = 1 \ (1 \leq j \leq p) \rangle.$$

The abelianized isoperimetric functions for G and K are equivalent [4], so in order to show that the abelianized isoperimetric function for G is at least exponential, it suffices to show that the abelianized isoperimetric function for K is at least exponential. We will replace G with K and ϕ with ϕ^k, then, so we may assume that $|\tilde{N} : \psi_2(\tilde{N})| > 2^c$.

Lemma 6.1 *There are elements $b_2, ..., b_{c+1} \in N$ such that the only element of the support of $(1 - b_{c+1})(1 - b_c) \cdots (1 - b_2)I(N')$ which lies in $\psi_2(\tilde{N})$ is $1I(N')$.*

(By the *support* of an element ϵ of a group ring, we mean those group elements which occur in the expression for ϵ with non-zero coefficient.)

Proof. Choose the b_i by induction on i. First, since ψ_2 is not onto, we can take b_2 to be any element of N for which $b_2 I(N') \notin \psi_2(\tilde{N})$. Now assume that $b_2, ..., b_{i-1}$ have been chosen so that among the products

$$\prod_{j \in I} b_j I(N'),$$

with $I \subseteq \{2, ..., i-1\}$, the only one of these products that is an element of $\psi_2(\tilde{N})$ is the empty product. There are 2^{i-1} such products. Since $|\tilde{N} : \psi_2(\tilde{N})| > 2^c > 2^{i-1}$, there must be an element b_i which does not lie in the same coset as any of the products above. The elements $b_2, ..., b_{c+1}$ chosen in this way have the required properties. □

Let F be the free group on the set $\{t, a_1, ..., a_l\}$. Let b_1 be the element a_h and let d be the element a_j in Proposition 4.5. Given any natural number n, let $w(n) = w \in F$ be the word

$$w := [b_1^{t^n}, b_2, ..., b_{c+1}].$$

Since N is nilpotent of class c and $b_i \in N$ for all i, the element w of F represents the element 1 in the group G, so w can be written as a product of conjugates of relators

$$w =_F \prod_{k=1}^{q(n)} (r_k^{\epsilon_k})^{v_k},$$

where each r_k is a relator, $\epsilon_k = \pm 1$, and $v_k \in F$. In order to find a lower bound for the abelianized isoperimetric function for G, as in [4] we employ the Fox free differential calculus. The Fox derivatives of both of the expressions above for w need to be computed and compared to get information on the size of $q(n)$.

Let δ be the composition of the Fox partial derivative with respect to the generator t, $\frac{\partial}{\partial t} : F \to \mathbf{Z}F$, with the canonical map $\mathbf{Z}F \to \mathbf{Z}G$. Then $\delta(t) = 1$ and $\delta(a_i) = 0$ for each i, and for any words u, v, we have

$$\begin{aligned}
\delta(uv) &= \delta(u) + u\delta(v), \\
\delta(u^{-1}) &= -u^{-1}\delta(u), \\
\delta(v^u) &= \delta(uvu^{-1}) = (1 - v^u)\delta(u) + u\delta(v), \text{ and} \\
\delta([u, v]) &= \delta(uvu^{-1}v^{-1}) = (1 - v^u)\delta(u) + (u - [u, v])\delta(v).
\end{aligned}$$

Applying these to the definition of w, we get

$$\delta(w) =_{\mathbf{Z}G} (1 - b_{c+1}^{[b_1^{t^n}, b_2, ..., b_c]})\delta([b_1^{t^n}, b_2, ..., b_c]) + ([b_1, ..., b_c] - 1)\delta(b_{c+1}).$$

Now $\delta(b_{c+1}) = 0$ because b_{c+1} is a product of the generators a_i, so the second term is trivial. Inductively, we get

$$\delta(w) =_{\mathbf{Z}G} (1 - b_{c+1}^{[b_1^{t^n}, b_2, ..., b_c]})(1 - b_c^{[b_1^{t^n}, b_2, ..., b_{c-1}]})...(1 - b_2^{b_1^{t^n}})(1 - b_1^{t^n})h_n(t),$$

where $h_n(t) =_{\mathbf{Z}G} 1 + t + t^2 + ... + t^{n-1}$.

Using the expression of w as a product of conjugates of relators, we get

$$\delta(w) =_{\mathbf{Z}G} \sum_{k=1}^{q} \epsilon_k v_k \delta(r_k).$$

The relators which include the generator t in their expression are of the form

$$R_i =_F a_i^t (x_i(a_1, ..., a_l))^{-1}.$$

Then when $r_k = R_i$, the Fox derivative $\delta(r_k) = \delta(R_i) = 1 - a_i^t$. For each $1 \leq i \leq l$, let $g_i(n) = g_i \in \mathbf{Z}G$ be the coefficient of $\delta(R_i)$ in the expression for $\delta(w)$. Then

$$\delta(w) =_{\mathbf{Z}G} \sum_{i=1}^{l} g_i(1 - a_i^t).$$

Setting these two expressions for $\delta(w)$ equal, we get

$$(1 - b_{c+1}^{[b_1^{t^n}, b_2, ..., b_c]})(1 - b_c^{[b_1^{t^n}, b_2, ..., b_{c-1}]})...(1 - b_2^{b_1^{t^n}})(1 - b_1^{t^n})h_n(t) =_{\mathbf{Z}G} \sum_{i=1}^{l} g_i(1 - a_i^t). \quad (1)$$

We can extract fom each side of this equality, those elements of the support which lie in N, together with their coefficients. Note that we have chosen each $b_i \in N$. This yields

$$(1 - b_{c+1}^{[b_1^{t^n}, b_2, ..., b_c]})(1 - b_c^{[b_1^{t^n}, b_2, ..., b_{c-1}]})...(1 - b_2^{b_1^{t^n}})(1 - b_1^{t^n}) =_{\mathbf{Z}N} \sum_{i=1}^{l} g_i'(1 - a_i^t) \quad (2)$$

with $g_i'(n) = g_i' \in \mathbf{Z}N$ equal to the $\mathbf{Z}N$ terms in g_i.

There is a natural homomorphism $\mathbf{Z}N \to \mathbf{Z}\tilde{N}$ induced by the map $N \to \tilde{N}$. The images of the the left and right hand sides of equation (2) under this map can be written

$$(1 - b_{c+1})(1 - b_c) \cdots (1 - b_2)(1 - b_1^{t^n}) =_{\mathbf{Z}\tilde{N}} \sum_{i=1}^{l} g_i'(1 - a_i^t) \quad (3)$$

where we denote cosets of $I(N')$ by the same letters as their representatives.

We next restrict this equation to terms that lie in the subgroup $\psi_2(\tilde{N})$ from Lemma 6.1. In this case, equation (3) becomes

$$(1 - b_1^{t^n}) =_{\mathbf{Z}\psi_2(\tilde{N})} \sum_{i=1}^{l} g_i''(1 - a_i^t), \quad (4)$$

with $g_i''(n) = g_i'' \in \mathbf{Z}\psi_2(\tilde{N})$ for each i.

Since $\tilde{N} = \mathbf{Z}a_1 \oplus \cdots \oplus \mathbf{Z}a_m$, there is a homomorphism $\mathbf{Z}\tilde{N} \to \mathbf{Z}\langle d \rangle$, sending a_i to 1 for each $i \neq j$, and sending $a_j = d$ to itself. Restrict this homomorphism to $\mathbf{Z}\psi_2(\tilde{N})$. Recall from Proposition 4.5 and the choice of b_1 and d that the exponent sum of the occurrences of d in $b_i^{t^n}$ is $\mu(n)$. In equation (4), this gives

$$(1 - d^{\mu(n)}) =_{\mathbf{Z}\langle d \rangle} \sum_{i=1}^{l} g_i'''(1 - d^{\nu_i}), \tag{5}$$

with $g_i'''(n) = g_i''' \in \mathbf{Z}\langle d \rangle$ and $\nu_i \in \mathbf{Z}$ is the coefficient $A_\phi(j, i)$. Because the group ring $\mathbf{Z}\langle d \rangle$ is an integral domain, we can divide through both sides by $1 - d$ and set $d = 1$; the latter is effected more formally by applying the augmentation map. This yields

$$\mu(n) =_{\mathbf{Z}} \sum_{i=1}^{l} g_i^{(4)} \nu_i, \tag{6}$$

with $g_i^{(4)}(n) = g_i^{(4)} \in \mathbf{Z}$.

Since $\mu(n)$ grows exponentially with n, for some i, $g_i^{(4)}(n)$ must also grow exponentially large with n. Now $g_i^{(4)}$ is constructed from the expression

$$w(n) =_F \prod_{k=1}^{q(n)} (r_k^{\epsilon_k})^{v_k}$$

of w as a product of conjugates of relators, by taking

$$g_i^{(4)} = \sum_k \epsilon_k,$$

where the sum ranges over all k such that $r_k = R_i$ and $v_k \in N$. Therefore, the number of summands of the form $\epsilon_k v_k \delta(r_k)$ in the expression for $\delta(w)$, and the number of factors of the form $(r_k^{\epsilon_k})^{v_k}$ in the expression for $w(n)$ as a product of conjugates of relators, is at least $\mu(n)$, which grows exponentially with n. Since this expression for $w(n)$ was arbitrary, the number of summands in any expression of $\delta(w)$ and the number of factors in any expression for $w(n)$ as a product of conjugates of relators must be at least $\mu(n)$ also. So the abelianized isoperimetric function (and hence also the Dehn function) for the group G in this case is bounded below by an exponential function.

This completes the proof of the Main Theorem.

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