

HOMOLOGY AND CLOSURE PROPERTIES OF AUTOSTACKABLE GROUPS

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ABSTRACT. Autostackability for finitely presented groups is a topological property of the Cayley graph combined with formal language theoretic restrictions, that implies solvability of the word problem. The class of autostackable groups is known to include all asynchronously automatic groups with respect to a prefix-closed normal form set, and all groups admitting finite complete rewriting systems. Although groups in the latter two classes all satisfy the homological finiteness condition FP_∞ , we show that the class of autostackable groups includes a group that is not of type FP_3 . We also show that the class of autostackable groups is closed under graph products and extensions.

1. INTRODUCTION

Autostackable groups are an extension of the notions of automatic groups and groups with finite complete rewriting systems, introduced by Holt and the first two authors in [6]. An autostackable structure for a finitely generated group implies a finite presentation, a solution to the word problem, a recursive algorithm for building van Kampen diagrams, and tame combability [5], [4]. Moreover, in contrast to automatic groups, the class of autostackable groups includes all fundamental groups of 3-manifolds with a uniform geometry [6].

Autostackability is a topological property of the Cayley graph, together with a language theoretic restriction on this property. More specifically, let G be a group with a finite inverse-closed generating set A , and let $\Gamma = \Gamma(G, A)$ be the associated Cayley graph. Denote the set of directed edges in Γ by \vec{E} , and the set of directed edge paths by \vec{P} . For each $g \in G$ and $a \in A$, let $e_{g,a}$ denote the directed edge with initial vertex g , terminal vertex ga , and label a ; we view the two directed edges $e_{g,a}$ and $e_{ga,a^{-1}}$ to have a single underlying undirected edge in Γ .

A *flow function* associated to a maximal tree \mathcal{T} in Γ is a function $\Phi : \vec{E} \rightarrow \vec{P}$ satisfying the properties that:

- (F1) For each edge $e \in \vec{E}$, the path $\Phi(e)$ has the same initial and terminal vertices as e .
- (F2d) If the undirected edge underlying e lies in the tree \mathcal{T} , then $\Phi(e) = e$.
- (F2r) The transitive closure $<_\Phi$ of the relation $<$ on \vec{E} defined by
 - $e' < e$ whenever e' lies on the path $\Phi(e)$ and the undirected edges underlying both e and e' do not lie in \mathcal{T} ,

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is a well-founded strict partial ordering.

The flow function is *bounded* if there is a constant k such that for all $e \in \vec{E}$, the path $\Phi(e)$ has length at most k . That is, the map Φ fixes the edges lying in the tree T and describes a “flow” of the non-tree edges toward the tree (or toward the basepoint); starting from a non-tree edge and iterating this function finitely many times results in a path in the tree.

In order to place a language theoretic restriction on Φ , we use functions that convert between paths and words. Define $\text{label} : \vec{P} \rightarrow A^*$ to be the function that maps each directed path to the word labeling that path. For each element $g \in G$, let y_g denote the label of the unique geodesic (i.e., without backtracking) path in the maximal tree \mathcal{T} from the identity element 1 of G to g , and let $\mathcal{N} = \mathcal{N}_{\mathcal{T}} := \{y_g \mid g \in G\}$ denote the set of these (unique) normal forms. Define $\text{path} : \mathcal{N} \times A^* \rightarrow \vec{P}$ by $\text{path}(y_g, w) :=$ the path in Γ that starts at g and is labeled by w .

Definition 1.1. [5, 6] *Let G be a group with a finite inverse-closed generating set A .*

- (1) *The group G is stackable over A if there is a bounded flow function on a maximal tree in the associated Cayley graph.*
- (2) *The group G is algorithmically stackable over A if G admits a bounded flow function Φ for which the graph*

$$\text{graph}(\phi) := \{(y_g, a, \text{label}(\Phi(\text{path}(y_g, a)))) \mid g \in G, a \in A\}$$

of the stacking map $\phi := \text{label} \circ \Phi \circ \text{path}$ is computable.

- (3) *The group G is autostackable over A if G has a bounded flow function Φ for which the graph of the associated stacking map is synchronously regular.*

A stackable group G over a finite generating set A is finitely presented, with finite presentation $R_{\Phi} = \langle A \mid \{\phi(y, a) = a \mid y \in \mathcal{N}_{\mathcal{T}}, a \in A\} \rangle$ associated to the flow function Φ . The set $\mathcal{N}_{\mathcal{T}}$ is a prefix-closed set of normal forms for G . A bounded flow function is equivalent to a bounded complete prefix-rewriting system for G over A , for which the irreducible words are exactly the elements of the set $\mathcal{N}_{\mathcal{T}}$. (See Section 2.2 for definitions of rewriting and prefix-rewriting systems.) Moreover, a group is autostackable if and only if it admits a synchronously regular bounded complete prefix-rewriting system. Algorithmic stackability (and hence also autostackability) implies a solution of the word problem; the set of rules of the associated prefix-rewriting system are computable, and give an algorithm to rewrite any word to the normal form representing the same group element. The class of autostackable groups includes all groups that are asynchronously automatic with respect to a prefix-closed set of (unique) normal forms, and all groups that admit a finite complete rewriting system. The class of stackable groups also includes all almost convex groups. For proofs of these and other results on autostackable groups, see [5] and [6].

Section 2 of this paper contains notation and definitions used throughout the paper, including background on language theory.

In Section 3, we show that the classes of autostackable, stackable, and algorithmically stackable groups are all closed under taking graph products (including free and direct products), extensions, and finite index supergroups (i.e., groups containing a finite index subgroup in the class). For the two properties that motivated autostackability, we note that

the class of groups admitting a finite complete rewriting system is closed under all three of these constructions ([16], [17], [15]), but the class of automatic groups is only closed under graph products and finite index supergroups ([16], [3]); in particular, a nilpotent group that is not virtually abelian is not automatic [11, Theorem 8.2.8]. The closure results in Section 3 show that any extension of a automatic group by another automatic group, such that the normal forms in both cases are prefix-closed (and unique), is autostackable.

The closure results of Section 3 leave open the question of whether the classes of autostackable, stackable, and algorithmically stackable groups are closed under taking finite index subgroups. Although it is known that finite index subgroups of automatic groups are automatic [3], it is an open question whether every finite index subgroup of a group with a finite complete rewriting system also has a finite complete rewriting system [8, p. 41], [15, Question 2], [23], or an autostackable structure.

In Section 4 we show that the class of autostackable groups includes groups with a wider range of homological finiteness properties than those of automatic groups or groups with finite complete rewriting systems. A group G has homological type FP_n if there is a partial projective resolution of length n , by finitely generated $\mathbb{Z}G$ -modules, of the module \mathbb{Z} (with trivial G action). In the case that G has type FP_n for all $n \in \mathbb{N}$, then G is said to be of type FP_∞ . Alonso [1] has shown that all groups that admit a bounded combing, including all automatic groups, have type FP_∞ . Groups with finite complete rewriting systems also are of type FP_∞ ; this has been shown with a variety of proofs in papers by Anick [2], Brown [7], Groves [14], Farkas [12], Kobayashi [20], and Lafont [21]; see Cohen's survey [8] for more details.

Stallings [25] showed that the group

$$G := \langle a, b, c, d, s \mid [a, c] = [a, d] = [b, c] = [b, d] = 1, [s, ab^{-1}] = [s, ac^{-1}] = [s, ad^{-1}] = 1 \rangle$$

does not have the finiteness property FP_3 . The results above show that this group cannot be automatic, nor can it admit a finite complete rewriting system. Moreover, Elder and the second author have shown that this group does not satisfy the almost convex property [9], nor the weaker minimally almost convex property [10], on this generating set. However, in Section 4, we show in Theorem 4.1 that this group is autostackable.

Corollary 4.2. *There is an autostackable group that does not satisfy the homological finiteness condition FP_3 .*

Stallings' group also provides an example of a group that cannot have a finite complete rewriting system, but does admit a bounded complete prefix-rewriting system.

2. NOTATION AND BACKGROUND

Throughout this paper, let G be a group with a finite inverse-closed generating set A . Also throughout the paper we assume that no element of a generating set represents the identity element of the group, and no two elements of a generating set represent the same element of the group.

A set \mathcal{N} of *normal forms* for G over A is a subset of A^* such that the restriction of the canonical surjection $\rho : A^* \rightarrow G$ to \mathcal{N} is a bijection. As in Section 1, the symbol y_g denotes

the normal form for $g \in G$. By slight abuse of notation, we use the symbol y_w to denote the normal form for $\rho(w)$ whenever $w \in A^*$.

Let 1 denote the identity of G , and let λ denote the empty word in A^* . For a word $w \in A^*$, we write w^{-1} for the formal inverse of w in A^* , and let $l(w)$ denote the length of the word w . For words $v, w \in A^*$, we write $v = w$ if v and w are the same word in A^* , and write $v =_G w$ if v and w represent the same element of G .

Given a word $w \in A^*$, let $\text{last}(w)$ denote the last letter in A of the word w ; in the case that $w = \lambda$ contains no letters, then we let $\text{last}(w) := \lambda$. For any subset $Z \subset A$, we use $\text{suf}_Z(w)$, to denote the maximal suffix of w that lies in Z^* ; here $\text{suf}_Z(w) := \lambda$ if w does not end with a letter in Z .

Let Γ be the Cayley graph of G with respect to A . If \mathcal{N} is a prefix-closed set of normal forms for G over A , then \mathcal{N} determines a maximal tree \mathcal{T} in Γ , namely the set of all (undirected) edges underlying edge paths in Γ starting at the vertex 1 and labeled by words in \mathcal{N} .

2.1. Formal language theory. A *language* over a finite set A is a subset of the set A^* of all finite words over A . The set A^+ denotes the language $A^* \setminus \{\lambda\}$ of all nonempty words over A .

The *regular* languages over A are the subsets of A^* obtained from the finite subsets of A^* using finitely many operations from among union, intersection, complement, concatenation ($S \cdot T := \{vw \mid v \in S \text{ and } w \in T\}$), and Kleene star ($S^0 := \{\lambda\}$, $S^n := S^{n-1} \cdot S$ and $S^* := \bigcup_{n=0}^{\infty} S^n$). The class of regular languages is closed under both image and preimage via monoid homomorphisms (see, for example, [18, Theorem 3.5]). The class of regular sets is also closed under quotients ([18, Theorem 3.6]); we write out a special case of this in the following lemma for use in later sections of this paper.

Lemma 2.1. [18, Theorem 3.6] *If A is a finite set, $L \subseteq A^*$ is a regular language, and $w \in A^*$, then the quotient language $L/w := \{x \in A^* \mid xw \in L\}$ is also a regular language.*

Let $\$$ be a symbol not contained in A . The set $A_n := (A \cup \{\$\})^n \setminus \{(\$, \dots, \$)\}$ is the *padded n -tuple alphabet* derived from A . For any n -tuple of words $u = (u_1, \dots, u_n) \in (A^*)^n$, write $u_i = a_{i,1} \cdots a_{i,j_i}$ with each $a_{i,m} \in A$ for $1 \leq i \leq n$ and $1 \leq m \leq j_i$. Let $M := \max\{j_1, \dots, j_n\}$, and define $\tilde{u}_i := u_i \$^{M-j_i}$, so that each of $\tilde{u}_1, \dots, \tilde{u}_n$ has length M . That is, \tilde{u}_i is a word over the alphabet $(A \cup \{\$\})^*$, and we can write $\tilde{u}_i = c_{i,1} \cdots c_{i,M}$ with each $c_{i,m} \in A \cup \{\$\}$. The word $\mu(u) := (c_{1,1}, \dots, c_{n,1}) \cdots (c_{1,M}, \dots, c_{n,M})$ is the *padded word* over the alphabet A_n induced by the n -tuple (u_1, \dots, u_n) in $(A^*)^n$.

A subset $L \subseteq (A^*)^n$ is called a *synchronously regular language* if the *padded extension* set $\mu(L) := \{\mu(u) \mid u \in L\}$ of padded words associated to the elements of L is a regular language over the alphabet A_n . Closure of the class of synchronously regular languages under finite unions and intersections follows from these closure properties for regular languages. The following two lemmas on synchronously regular languages will also be used in later sections.

Lemma 2.2. [6, Lemma 2.3] *If L_1, \dots, L_n are regular languages over A , then their Cartesian product $L_1 \times \cdots \times L_n \subseteq (A^*)^n$ is synchronously regular.*

Lemma 2.3. [11, Theorem 1.4.6] *If $L \subset (A^*)^n$ is a synchronously regular language, then the projection on the first coordinate given by the set $\text{proj}_1(L) := \{u \mid \exists(u, u_2, \dots, u_n) \in L\}$ is a regular language over A .*

See [11] and [18] for more information about regular and synchronously regular languages.

2.2. Rewriting systems. The definitions and results in this section can be found in the text [24] by Sims.

A *complete rewriting system* for a group G consists of a set A and a set of “rules” $R \subseteq A^* \times A^*$ (with each $(u, v) \in R$ written $u \rightarrow v$) such that G is presented as a monoid by $G = \text{Mon}\langle A \mid u = v \text{ whenever } u \rightarrow v \in R \rangle$, and the rewritings of the form $xuy \rightarrow xvy$ for all $x, y \in A^*$ and $u \rightarrow v$ in R , with transitive closure \rightarrow^* , satisfy:

- (1) There is no infinite chain $w \rightarrow x_1 \rightarrow x_2 \rightarrow \dots$ of rewritings.
- (2) Whenever there is a pair of rules of the form $rs \rightarrow v$ and $st \rightarrow w$ [respectively, $s \rightarrow v$ and $rst \rightarrow w$] in R with $r, s, t, v, w \in A^*$ and $s \neq \lambda$, then there are rewritings $vt \rightarrow^* z$ and $rw \rightarrow^* z$ [respectively, $rvt \rightarrow^* z$ and $w \rightarrow^* z$] for some $z \in A^*$.

The rewriting system is *finite* if the sets A and R are both finite.

The pairs of rules in item (2) are called *critical pairs*, and when property (2) holds, the critical pairs are said to be *resolved*. The set $\text{lrr}(R)$ of irreducible words (that is, words that cannot be rewritten) is a set of normal forms for the group G presented by the complete rewriting system.

A *complete prefix-rewriting system* for a group G consists of a set A and a set of rules $R \subseteq A^* \times A^*$ (with each $(u, v) \in R$ written $u \rightarrow v$) such that G is presented (as a monoid) by $G = \text{Mon}\langle A \mid u = v \text{ whenever } u \rightarrow v \in R \rangle$, and the rewritings $uy \rightarrow vy$ for all $y \in A^*$ and $u \rightarrow v$ in R satisfy: (1) There is no infinite chain $w \rightarrow x_1 \rightarrow x_2 \rightarrow \dots$ of rewritings, and (2) each $g \in G$ is represented by exactly one irreducible word over A . (The difference between a prefix-rewriting system and a rewriting system is that rewritings of the form $xuy \rightarrow xvy$ with $x \in A^* \setminus \{\lambda\}$ and $u \rightarrow v \in R$ are allowed in a rewriting system, but only rewritings $uy \rightarrow vy$ are allowed in a prefix-rewriting system.) The prefix-rewriting system is *bounded* if A is finite and there is a constant k such that for each pair (u, v) in R there are words $s, t, w \in A^*$ such that $u = ws$, $v = wt$, and $l(s) + l(t) \leq k$.

3. CLOSURE PROPERTIES OF AUTOSTACKABLE GROUPS

3.1. Graph products.

In this section we prove the first of the closure properties, that each of the stackability properties is preserved by the graph product construction.

Given a finite simplicial graph Λ (with no loops or multiple edges) with vertices v_1, \dots, v_n , such that each vertex v_i is labeled by a group G_i , the associated *graph product* is the quotient $G\Lambda$ of the free product of the groups G_i by the relations that elements of vertex groups corresponding to adjacent vertices in Λ commute. Special cases include the free product (if Λ is totally disconnected) and direct product (if Λ is complete) of the groups G_i .

For each $1 \leq i \leq n$, let A_i be a finite inverse-closed generating set for the vertex group G_i . In this section we use the generating set $A := \cup_{i=1}^n A_i$ of $G\Lambda$ for our constructions. For each i , we let $I_i \subseteq \{1, \dots, n\}$ denote the set of indices k such that v_k and v_i are adjacent in Λ . This set I_i can be partitioned into the subsets $I_i^> := I_i \cap \{i+1, \dots, n\}$ and $I_i^< := I_i \cap \{1, \dots, i-1\}$. Let $C_i := A_i \cup \{\mathbb{I}, \succ\}$, where \mathbb{I} and \succ denote distinct letters not in A , and define a monoid homomorphism $\pi_i : A^* \rightarrow C_i^*$ by defining

$$\pi_i(a) := \begin{cases} a & \text{if } a \in A_i \\ \succ & \text{if } a \in A_k \text{ for some } k \in I_i^> \\ \lambda & \text{if } a \in A_k \text{ for some } k \in I_i^< \\ \mathbb{I} & \text{if } a \in A_k \text{ for some } k \in I \setminus (I_i \cup \{i\}). \end{cases}$$

Lemma 3.1. *Let $G\Lambda$ be a graph product of the groups $G_i = \langle A_i \rangle$, let $A := \cup_{i=1}^n A_i$, and suppose that for each index i the set \mathcal{N}_i is a prefix-closed set of normal forms for G_i over the generators A_i . Then the language*

$$\mathcal{N}_\Lambda := \cap_{i=1}^n \pi_i^{-1}((\mathcal{N}_i \succ^* \mathbb{I})^* \mathcal{N}_i \succ^*)$$

is a prefix-closed set of normal forms for $G\Lambda$.

Proof. Over the larger generating set $X := \cup_{i=1}^n X_i$ of $G\Lambda$ where each $X_i := G_i \setminus \{1_{G_i}\}$, we note that the set of rules $R := R' \cup R''$, where $R' := \{gh \rightarrow (gh) \mid g, h \in X_i, i \in \{1, \dots, n\}\}$ and $R'' := \{gwh \rightarrow hgw \mid g \in X_i, h \in X_j, j \in I_i^<, w \in (\cup_{k \in I_j} X_k)^*\}$, is a complete rewriting system for G . Here (gh) denotes the element of X_i corresponding to the product gh in G_i if $gh \neq_{G_i} 1_{G_i}$, and (gh) denotes the empty word λ if $gh =_{G_i} 1_{G_i}$. Indeed, if we let $S := \{s_1, \dots, s_n\}$ have the total ordering defined by $s_i < s_j$ whenever $i < j$, and define the monoid homomorphism $\alpha : X^* \rightarrow S^*$ by $\alpha(g) := s_i$ for each $g \in X_i$, then each rewriting $xuy \rightarrow xvy$ with $x, y \in X^*$ and $u \rightarrow v \in R$ satisfies the property that $\alpha(xuy) >_{sl} \alpha(xvy)$, where $>_{sl}$ is the (well-founded) shortlex ordering on S^* , and so there cannot be an infinite sequence of rewritings. It is also straightforward to check that the critical pairs are resolved (see Section 2.2 for this terminology), and so this is a complete rewriting system. Hence the set $\text{lrr}(R)$ of irreducible words for this system is a set of normal forms for $G\Lambda$ over X .

Now let $\beta : X^* \rightarrow A^*$ be the monoid homomorphism mapping each $g \in X_i$ to the normal form $\beta(g)$ of g in \mathcal{N}_i . Then $\beta(\text{lrr}(R))$ is a set of normal forms for $G\Lambda$ over A .

Given a word $w \in \text{lrr}(R)$, the image $\pi_i(\beta(w))$ lies in $(\mathcal{N}_i \succ^* \mathbb{I})^* \mathcal{N}_i \succ^*$ for all i , by the choice of the rewriting rules in R ; hence, $\beta(\text{lrr}(R)) \subseteq \mathcal{N}_\Lambda$. In the other direction, for any word $x \in \mathcal{N}_\Lambda$, we can consider the word x as an element of X^* using the inclusion of A in X . Since $\pi_i(x)$ lies in $(\mathcal{N}_i \succ^* \mathbb{I})^* \mathcal{N}_i \succ^*$ for each i , the only rules of the rewriting system R that can be applied to x are in the set R' . Since each element of G_i is represented by only one word in \mathcal{N}_i , and the normal form set \mathcal{N}_i is prefix-closed, it follows that nonempty subwords of words in \mathcal{N}_i cannot represent the trivial element of G_i . Consequently any sequence of rewritings of x using the rules in R' may only replace words in X_i^+ (that is, nonempty words over X_i) with words again in X_i^+ . Hence any further rewritings from the system R again can only apply rules in R' , resulting in an irreducible word x' . Applying β returns the original word $\beta(x') = x$. Therefore $\mathcal{N}_\Lambda = \beta(\text{lrr}(R))$.

Finally, prefix-closure of the sets \mathcal{N}_i yields prefix-closure of the languages $(\mathcal{N}_i \succ^* \mathbb{I})^* \mathcal{N}_i \succ^*$ for each i , which in turn implies prefix-closure of the language \mathcal{N}_Λ . \square

We note that the normal forms $\text{lrr}(R)$ in Lemma 3.1 are the same as those developed by Green in [13], and the set \mathcal{N}_Λ is also constructed using alternative methods in [16] and [19]. Next we use the normal form set \mathcal{N}_Λ to prove the closure properties for graph products.

Theorem 3.2. *For $1 \leq i \leq n$ let G_i be an autostackable [respectively, stackable, algorithmically stackable] group on a finite inverse-closed generating set A_i . Then any graph product $G\Lambda$ of these groups with the generating set $A := \cup_{i=1}^n A_i$ is also autostackable [respectively, stackable, algorithmically stackable].*

Proof. Let \mathcal{N}_i , Φ_i , and ϕ_i be the normal form set over A_i , the bounded flow function, and the stacking map for the group G_i , respectively. To streamline the discussion, for each $1 \leq i \leq n$ we denote the language $(\mathcal{N}_i \succ^* \mathbb{I})^* \mathcal{N}_i \succ^*$ by \mathcal{L}_i . Also let \mathcal{N}_Λ be the normal form set for $G\Lambda$ from Lemma 3.1, and as usual denote the normal form for $g \in G\Lambda$ by y_g . Let Γ be the Cayley graph of $G\Lambda$ over A , with sets \vec{E} of directed edges and \vec{P} of directed edge paths, and let \mathcal{T} be the maximal tree in Γ corresponding to this set of normal forms.

Step 1: Stackable.

We begin by defining a function $\phi : \mathcal{N}_\Lambda \times A \rightarrow A^*$ as follows. Recall from Section 2 that for any word $w \in A^*$, $\text{last}(w)$ denotes the last letter of the word w , and $\text{suf}_{A_i}(w)$, which we shorten to $\text{suf}_i(w)$ throughout this proof, denotes the maximal suffix of w in the letters of the subset A_i of A . Now for each $y_g \in \mathcal{N}_\Lambda$ and $a \in A_k$ we define

$$\phi(y_g, a) := \begin{cases} \phi_k(\text{suf}_k(y_g), a) & \text{if } \pi_k(y_g) \notin C_k^* \succ \\ \text{last}(y_g)^{-1} a \text{last}(y_g) & \text{if } \pi_k(y_g) \in C_k^* \succ. \end{cases}$$

We also let $\Phi : \vec{E} \rightarrow \vec{P}$ denote the function $\Phi(e_{g,a}) := \text{path}(g, \phi(y_g, a))$.

It follows immediately from the definition of Φ that property (F1) of the definition of flow function holds.

To check property (F2d), consider any directed edge $e = e_{g,a}$ whose underlying undirected edge lies in the tree \mathcal{T} . Then either $y_g a$ or $y_g a^{-1}$ is an element of \mathcal{N}_Λ . Let k be the index such that $a \in A_k$. Now either $\pi_k(y_g a) = [\pi_k(y_g)]a \in [(\mathcal{N}_k \succ^* \mathbb{I})^* \mathcal{N}_k \succ^*]a \subseteq \mathcal{L}_k$ or $\pi_k(y_g a^{-1}) = \pi_k(y_g) \in \mathcal{L}_k a^{-1} \cap \mathcal{L}_k$, and so in both cases we have $\pi_k(y_g) \notin C_k^* \succ$. Hence $\phi(y_g, a) = \phi_k(\text{suf}_k(y_g), a)$. Write $\pi_k(y_g) = vw$ where $v \in (\mathcal{N}_k \succ^* \mathbb{I})^*$ and $w \in \mathcal{N}_k$. Since $\pi_k(b) = \lambda$ for all $b \in \cup_{i \in I_k^<} A_i$, then in the word y_g , the letters from w may be interspersed with, or precede, such a letter b . However, if $w \neq \lambda$ and b is the first letter in $\cup_{i \in I_k^<} A_i$ occurring after the first letter of w in y_g , and we let a' be the letter from w that immediately precedes b in y_g , then for the index i such that $b \in A_i$, the word $\pi_i(a'b) = \succ b$ is a subword of $\pi_i(y_g)$, giving a contradiction. This shows that $\text{suf}_k(y_g) = w$. Now we have that either $y_g a \in \mathcal{N}_\Lambda$, in which case $\text{suf}_k(y_g) a = wa \in \mathcal{N}_k$, or else $y_g = y_g a^{-1}$, in which case $\text{suf}_k(y_g)$ ends with the letter a^{-1} . Since the flow function Φ_k satisfies property (F2d), then $\phi_k(\text{suf}_k(y_g), a) = a$. Therefore $\Phi(e) = e$, and (F2d) holds for Φ .

Next we turn to property (F2r). We define a function $\psi : \vec{E} \rightarrow \mathbb{N}^2$ as follows. Let $e_{g,a} \in \vec{E}$, and let k be the index such that $a \in A_k$. Define $\psi(e_{g,a}) := (l(y_g), 0)$ if $\pi_k(y_g) \in (C_k^*)^* \succ$, and $\psi(e_{g,a}) := (0, \text{dcl}_k(\text{suf}_k(y_g), a))$ if $\pi_k(y_g) \notin (C_k^*)^* \succ$, where the *descending chain length* $\text{dcl}_k(w, a)$ denotes the maximum length of a descending chain $e_{w,a} >_{\Phi_k} e_1 >_{\Phi_k} e_2 \cdots >_{\Phi_k} e_n$ of edges for the well-founded ordering obtained from the flow function Φ_k . Let $<_{\mathbb{N}^2}$ be the lexicographic ordering on \mathbb{N}^2 ; that is, $(a, b) <_{\mathbb{N}^2} (c, d)$ if either $a < c$ or $a = c$ and $b < d$, where we use the standard ordering on \mathbb{N} . Note that $<_{\mathbb{N}^2}$ is a well-founded strict partial ordering. In order to show that property (F2r) holds for the function Φ , it suffices to show that whenever $e' <_{\Phi} e$, then $\psi(e') <_{\mathbb{N}^2} \psi(e)$. Making use of the fact that the ordering $<_{\Phi}$ is a transitive closure of another relation, it then suffices to show that whenever e' is an edge of $\Phi(e)$ and neither e' nor e is in the tree \mathcal{T} , then $\psi(e') <_{\mathbb{N}^2} \psi(e)$.

Now suppose that $e = e_{g,a}$ is any element of \vec{E} and the undirected edge of Γ underlying $e_{g,a}$ does not lie in \mathcal{T} , and let k be the index such that $a \in A_k$. In this case the words $y_g a$ and $y_g a^{-1}$ are not in the normal form set \mathcal{N}_{Λ} . Note that for any index $i \neq k$, the image of $y_g a$ under π_i has the form $\pi_i(y_g a) = \pi_i(y_g)\pi_i(a)$ where $\pi_i(y_g) \in \mathcal{L}_i$ and $\pi_i(a) \in \{\succ, \lambda, \mathbb{I}\}$, and therefore also $\pi_i(y_g a) \in \mathcal{L}_i$. Then since $y_g a \notin \mathcal{N}_{\Lambda}$, we must have $\pi_k(y_g a) = \pi_k(y_g)a \notin \mathcal{L}_k = (\mathcal{N}_k \succ^* \mathbb{I})^* \mathcal{N}_k \succ^*$. We treat the cases in which $\pi_k(y_g)$ does or does not end with the letter \succ separately.

Suppose first that $\pi_k(y_g)$ does not end with \succ . Then $\pi_k(y_g) \in (\mathcal{N}_k \succ^* \mathbb{I})^* \mathcal{N}_k$; write $\pi_k(y_g) = uv$ for $u \in (\mathcal{N}_k \succ^* \mathbb{I})^*$ and $v \in \mathcal{N}_k$. Applying the discussion in the proof of (F2d) above, then $v = \text{suf}_k(y_g)$; hence we can write $y_g = \tilde{u}v$. Now $\psi(e) = (0, \text{dcl}_k(\text{suf}_k(y_g), a)) = (0, \text{dcl}_k(v, a))$ where $\text{dcl}_k(v, a)$ is the maximum length of a descending chain of edges starting from $e_{v,a}$ for the flow function Φ_k , and since $\pi_k(y_g a) \notin \mathcal{L}_k$ and $\pi_k(y_g)$ does not end with a^{-1} , the descending chain length satisfies $\text{dcl}_k(v, a) > 0$. For any edge e' in the path $\Phi(e)$, we have $e' = e_{\tilde{u}v',c}$ for some $v' \in \mathcal{N}_k$ and $c \in A_k$ such that $e_{v',c}$ is an edge of the path $\Phi_k(e_{v,a})$. Note that, as above, $\text{suf}_k(\tilde{u}v') = v'$. In the case that e' does not lie in the tree \mathcal{T} , then $e_{v',c} <_{\Phi_k} e_{v,a}$ and $\pi_k(\tilde{u}v') = \pi_k(\tilde{u})v' \in \mathcal{L}_k \cap (C_k^* \setminus C_k^* \succ)$, and hence $\psi(e') = (0, \text{dcl}_k(\text{suf}_k(y_{\tilde{u}v'}), c)) = (0, \text{dcl}_k(v', c)) <_{\mathbb{N}^2} (0, \text{dcl}_k(v, a)) = \psi(e)$.

On the other hand suppose that $\pi_k(y_g)$ ends with \succ . The path $\Phi(e)$ contains three directed edges $e_1 := e_{g,b^{-1}}$, $e_2 := e_{gb^{-1},a}$, and $e_3 := e_{gb^{-1}a,b}$ where $b := \text{last}(y_g)$. Note that $y_g = y_{gb^{-1}}b$, and $l(y_g) \geq 1$. The undirected edge underlying e_1 lies in the tree \mathcal{T} . If e_2 does not lie in the tree \mathcal{T} , then $y_{gb^{-1}}a \notin \mathcal{N}_{\Lambda}$, and so the argument (two paragraphs) above also shows that $\pi_k(y_{gb^{-1}}a) \notin \mathcal{L}_k$. In that case, either $\pi_k(y_{gb^{-1}})$ ends with the letter \succ , in which case $\psi(e_2) = (l(y_{gb^{-1}}), 0) = (l(y_g) - 1, 0) <_{\mathbb{N}^2} (l(y_g), 0) = \psi(e)$, or else $\pi_k(y_{gb^{-1}}a)$ does not end with \succ , in which case $\psi(e_2) = (0, \text{dcl}_k(\text{suf}_k(y_{gb^{-1}}), a)) <_{\mathbb{N}^2} (l(y_g), 0) = \psi(e)$. So in all cases, $\psi(e_2) <_{\mathbb{N}^2} \psi(e)$.

Finally, we show that edge e_3 lies in the tree \mathcal{T} . First note that since $\pi_k(y_g) \in (\mathcal{N}_k \succ^* \mathbb{I})^* \mathcal{N}_k \succ^+$, we can decompose the normal form for g as $y_g = u_1 u_2 u_3 b$ where $\pi_k(u_1) \in (\mathcal{N}_k \succ^* \mathbb{I})^*$ and the length of the prefix u_1 is maximal with respect to this property, $u_2 \in \mathcal{N}_k$ (including the possibility that u_2 is the empty word λ), $\pi_k(u_3 b) \in \succ^+$, and the first letter c of $u_3 b$ satisfies $\pi_k(c) = \succ$. Note that $gb^{-1}ab =_{G\Lambda} ga =_{G\Lambda} u_1 y_{u_2 a} u_3 b$; we claim that $u_1 y_{u_2 a} u_3 b \in \mathcal{N}_{\Lambda}$. Indeed, for the index k we have $\pi_k(u_1 y_{u_2 a} u_3 b) = \pi_k(u_1) y_{u_2 a} \pi_k(u_3 b) \in (\mathcal{N}_k \succ^* \mathbb{I})^* \mathcal{N}_k \succ^+ \subseteq \mathcal{L}_k$. For each index $i \neq k$, the word $\pi_i(u_1 y_{u_2 a} u_3 b)$ is obtained from

$\pi_i(y_g)$ by removal of the subword $\pi_i(u_2)$ and insertion of the subword $\pi_i(y_{u_2a})$; we consider these steps separately. The word $\pi_i(u_1u_3b)$ is obtained from $\pi_i(y_g)$ by removing a subword from $\succ^* \cup \mathbb{I}^+$. Deleting a \succ^* subword preserves membership in the language \mathcal{L}_i . In the case that $\pi_i(u_2) \in \mathbb{I}^+$, then the vertices v_i and v_k of Λ are not adjacent, and $\pi_k(d) = \mathbb{I}$ for all $d \in A_i$; hence no letter of u_3b lies in A_i . Removal of a \mathbb{I}^+ subword from a suffix in $\{\succ, \mathbb{I}\}^*$ also preserves membership in \mathcal{L}_i . Hence we have $\pi_i(u_1u_3b) \in \mathcal{L}_i$. Next, $\pi_i(u_1y_{u_2a}u_3b)$ is obtained from $\pi_i(u_1u_3b)$ by the insertion of a subword from $\succ^* \cup \mathbb{I}^+$, and insertion of such a subword preserves membership in \mathcal{L}_i unless the inserted word is nonempty and immediately precedes a letter lying in A_i . In the case that $\pi_i(y_{u_2a}) \in \succ^+$, we have $i < k$. Moreover, the first letter c of u_3b satisfies $\pi_k(c) = \succ$, and so $c \in A_l$ for an index l satisfying $k < l$. Then $i < l$, and so the word $\pi_i(y_{u_2a})$ is inserted immediately before the letter $\pi_i(c) \in \{\succ, \mathbb{I}\}$. In the case that $\pi_i(y_{u_2a}) \in \mathbb{I}^+$, again the vertices v_i and v_k are not adjacent and no letter of A_i appears in u_3b , so no \mathbb{I} is inserted preceding a letter of A_i . Thus $\pi_i(u_1y_{u_2a}u_3b) \in \mathcal{L}_i$, completing the proof of the claim. Since the normal form $y_{gb^{-1}ab} = u_1y_{u_2a}u_3b$ ends with the letter b , then the edge e_3 lies in \mathcal{T} .

Thus we have that for any edge e' in the path $\Phi(e)$, either e' lies in the tree \mathcal{T} , or else $\psi(e') <_{\mathbb{N}^2} \psi(e)$, as required. This completes the proof of (F2r) and the proof that Φ is a flow function associated to the tree \mathcal{T} in the Cayley graph for $G\Lambda$ over A . It follows from the construction of Φ that this flow function is bounded by the constant $\max\{3, \text{bd}(\Phi_1), \dots, \text{bd}(\Phi_n)\}$, where for each index i , $\text{bd}(\Phi_i)$ denotes the bound on the flow function Φ_i .

Step 2: Autostackable.

Now suppose further that the set $\text{graph}(\phi_k)$ is synchronously regular for all indices $1 \leq k \leq n$. Following the piecewise description of the stacking function ϕ associated to Φ given in Step 1 above, its graph can be written in the form

$$\begin{aligned} \text{graph}(\phi) &= \bigcup_{k=1}^n \bigcup_{a \in A_k} [(\bigcup_{x \in \text{im}(\phi_k)} \text{graph}(\phi) \cap (A^* \times \{a\} \times \{x\})) \\ &\quad \bigcup (\bigcup_{i \in I_k} \bigcup_{b \in A_i} \text{graph}(\phi) \cap (A^* \times \{a\} \times \{b^{-1}ab\}))] \\ &= (\bigcup_{k \in \{1..n\}, a \in A_k, x \in \{\phi_k(y,a) \mid y \in \mathcal{N}_k\}} L_{k,a,x} \times \{a\} \times \{x\}) \\ &\quad \bigcup (\bigcup_{k \in \{1..n\}, a \in A_k, i \in I_k, b \in A_i} L'_{k,a,b} \times \{a\} \times \{b^{-1}ab\}) \end{aligned}$$

where

$$\begin{aligned} L_{k,a,x} &= \{y_g \in \mathcal{N}_\Lambda \mid \pi_k(y_g) \notin C_k^* \succ \text{ and } \phi_k(\text{suf}_k(y_g), a) = x\}, \text{ and} \\ L'_{k,a,b} &= \{y_g \in \mathcal{N}_\Lambda \mid \pi_k(y_g) \in C_k^* \succ \text{ and } \text{last}(y_g) = b\} \end{aligned}$$

Since the class of synchronously regular languages is closed under finite unions, and all finite sets are regular, then Lemma 2.2 shows that in order to prove that $\text{graph}(\phi)$ is synchronously regular, it suffices to show that the languages $L_{k,a,x}$ and $L'_{k,a,b}$ are regular.

We begin by considering the language \mathcal{N}_Λ of normal forms. Lemma 2.3 says that the projection of $\text{graph}(\phi_k)$ on the first coordinate, which is the language \mathcal{N}_k of normal forms for the group G_k , is regular. Since regular languages are closed under concatenation and Kleene star, then $\mathcal{L}_k = (\mathcal{N}_k \succ^* \mathbb{I})^* \mathcal{N}_k \succ^*$ is also regular. Finally closure of regular languages under homomorphic inverse image and finite intersection implies that \mathcal{N}_Λ is also regular.

Now $L'_{k,a,b} = \mathcal{N}_\Lambda \cap \pi_k^{-1}(C_k^* \succ) \cap A^*b$. Since the class of regular languages is closed under Kleene star, concatenation, and homomorphic preimage, then each of the three sets in this intersection is regular, and so $L'_{k,a,b}$ is a regular language.

The proof in Step 1 above shows that for any $y_g \in L_{k,a,x} \subseteq \mathcal{N}_\Lambda \cap \pi_k^{-1}(C_k^* \setminus C_k^* \succ)$, the word y_g can be written in the form $y_g = y' \text{suf}_k(y_g)$ where $\pi_k(y') \in (\mathcal{N}_k \succ^* \mathbb{I})^*$. Then $L_{k,a,x} = \mathcal{N}_\Lambda \cap \pi_k^{-1}((\mathcal{N}_k \succ^* \mathbb{I})^*) N_{k,a,x}$ where $N_{k,a,x} := \text{proj}_1(\text{graph}(\phi_k) \cap (A_k^* \times \{a\} \times \{x\}))$ is the set of all words $w \in \mathcal{N}_k$ such that $\phi_k(w, a) = x$. Since $\text{graph}(\phi_k)$ is synchronously regular and the intersection of two synchronously regular languages is synchronously regular, then applying Lemma 2.3 shows that the language $N_{k,a,x}$ is regular. Then using closure properties of regular languages, we also have that $L_{k,a,x}$ is regular.

Therefore the set $\text{graph}(\phi)$ is synchronously regular, and $G\Lambda$ is autostackable over A .

Step 3: Algorithmically stackable.

The proof in this case is similar to the argument in Step 2. \square

3.2. Extensions.

We continue the investigation into closure properties with the extension of a group K by a group Q .

Theorem 3.3. *Let $1 \rightarrow K \xrightarrow{i} G \xrightarrow{q} Q \rightarrow 1$ be a short exact sequence of groups and group homomorphisms. If K and Q are autostackable [respectively, stackable, algorithmically stackable] groups on finite inverse-closed generating sets A and B , respectively, and $\widehat{B} \subseteq G$ is an inverse-closed subset of G that bijects via q to B , then the group G with the generating set $i(A) \cup \widehat{B}$ is also autostackable [respectively, stackable, algorithmically stackable].*

Proof. Let \mathcal{N}_K , Φ_K , and ϕ_K be the normal form set, bounded flow function, and associated stacking map for K over A , and similarly let \mathcal{N}_Q , Φ_Q , and ϕ_Q be the normal form set, bounded flow function, and associated stacking map for Q over B . Let $K = \langle A | R \rangle$ and $Q = \langle B | S \rangle$ be the finite presentations obtained from these flow functions. By slight abuse of notation, we will consider the homomorphism i to be an inclusion map, and $A, K \subseteq G$, so that we may omit writing $i(\cdot)$. Let $C := A \cup \widehat{B}$.

For each $b \in B$, there is a unique element $\hat{b} \in \widehat{B}$ with $q(\hat{b}) = b$. For each word $w = b_1 \cdots b_n \in B^*$, we define $\text{hat}(w) := \hat{b}_1 \cdots \hat{b}_n$. Let $\bar{\cdot}$ be the map from \widehat{B}^* to B^* that reverses the map $\text{hat} : B^* \rightarrow \widehat{B}^*$; that is, for any letter $c \in \widehat{B}$, $\bar{c} = q(c) \in B$, and for any word $v = c_1 \cdots c_n \in \widehat{B}^*$, then $\bar{v} = \bar{c}_1 \cdots \bar{c}_n$. Define

$$\mathcal{N}_G := \mathcal{N}_K \text{hat}(\mathcal{N}_Q).$$

Since $Q \cong G/K$ and the set $\text{hat}(\mathcal{N}_Q) \subset \widehat{B}^*$ bijects (via q) to Q , the language $\text{hat}(\mathcal{N}_Q)$ is a set of coset representatives for G/K , and every element g of G can be written uniquely in the form $g = kp$ for some $k \in K$ and $p \in \text{hat}(\mathcal{N}_Q)$; that is, the set \mathcal{N}_G is a set of normal forms for G over the finite inverse-closed generating set C .

Let Γ be the Cayley graph of G with respect to C , and let \vec{E} and \vec{P} be the sets of directed edges and directed paths in Γ . Since both \mathcal{N}_K and \mathcal{N}_Q are prefix-closed, the language \mathcal{N}_G is prefix-closed as well, and so \mathcal{N}_G determines a maximal tree \mathcal{T} in Γ .

Step 1: Stackable.

For each $g \in G$, let y_g denote the normal form of g in \mathcal{N}_G . We also use the notation $y_g = u_g t_g$ where $u_g \in \mathcal{N}_K$ and $t_g \in \text{hat}(\mathcal{N}_Q)$. Note that G has the presentation

$$G = \langle C \mid R \cup \{\text{hat}(s) = u_{\text{hat}(s)} \mid s \in S\} \cup \{\hat{b}a = u_{\hat{b}a} \hat{b}^{-1} \hat{b} \mid a \in A, \hat{b} \in \widehat{B}\} \rangle;$$

in this step we use the relations from this presentation to construct a flow function for G over C .

Define a function $\phi : \mathcal{N}_G \times C \rightarrow C^*$ by

$$\phi(y_g, c) = \begin{cases} \phi_K(y_g, c) & \text{if } c \in A \text{ and } y_g \in A^* \\ \text{last}(y_g)^{-1} u_{\text{last}(y_g)} \text{clast}(y_g)^{-1} \text{last}(y_g) & \text{if } c \in A \text{ and } y_g \notin A^* \\ u_{c(\text{hat}(\phi_Q(q(g), q(c))))^{-1}} \text{hat}(\phi_Q(q(g), q(c))) & \text{if } c \in \widehat{B}. \end{cases}$$

Also define $\Phi : \vec{E} \rightarrow \vec{P}$ by $\Phi(e_{g,c}) := \text{path}(y_g, \phi(y_g, c))$.

It follows immediately from the definition of Φ and the presentation of G that property (F1) of the definition of flow function holds. To check property (F2d), suppose that $e = e_{g,c}$ is any edge in the tree \mathcal{T} ; then either $y_g c$ or $y_g c^{-1}$ lies in \mathcal{N}_G . If $c \in A$, then the definition of \mathcal{N}_G implies that $y_g \in A^*$, and either $y_g c$ or $y_g c^{-1}$ lies in \mathcal{N}_K . Then $\phi_G(y_g, c) = \phi_K(y_g, c) = c$, and so $\Phi(e) = e$. If instead $c \in \widehat{B}$, then either $u_g t_g c$ or $u_g t_g c^{-1}$ lies in \mathcal{N}_G . Now $\bar{t}_g \in \mathcal{N}_Q$, $\bar{c} \in B$, and either $\bar{t}_g \bar{c}$ or $\bar{t}_g \bar{c}^{-1}$ lies in \mathcal{N}_Q . Hence property (F2d) for Φ_Q implies that $\phi_Q(q(g), q(c)) = \phi_Q(\bar{t}_g, \bar{c}) = \bar{c}$, and so $\text{hat}(\phi_Q(q(g), q(c))) = c$. Note that $u_{cc^{-1}} = u_1 = \lambda$. Then again we have $\phi(y_g, c) = c$ and $\Phi(e) = e$. Hence property (F2d) holds for Φ .

Our procedure to check property (F2r) for Φ will again (as in the proof of Theorem 3.2) make use of a function $\psi : \vec{E} \rightarrow \mathbb{N}^2$ that captures information from property (F2r) for the flow functions Φ_K and Φ_Q . Let $\vec{E}_K = \{e_{k,a}^K \mid k \in K, a \in A\}$ be the set of directed edges of the Cayley graph of K over A , and define $\text{dcl}_K : \vec{E}_K \rightarrow \mathbb{N}$ by

$$\text{dcl}_K(e_{k,a}^K) = \text{maximal length of a descending chain } e_{k,a}^K >_{\Phi_K} e' >_{\Phi_K} e'' \cdots$$

Similarly let $\vec{E}_Q = \{e_{q,b}^Q \mid q \in Q, b \in B\}$ be the set of directed edges of the Cayley graph of Q over B , and let $\text{dcl}_Q(e_{q,b}^Q)$ be the maximum length of a descending chain starting at $e_{q,b}^Q$ for the well-founded strict partial ordering $>_{\Phi_Q}$. Now define $\psi : \vec{E} \rightarrow \mathbb{N}^2$ on the directed edges of the Cayley graph of G over C by

$$\psi(e_{g,c}) = \begin{cases} (1, \text{dcl}_K(e_{g,c}^K)) & \text{if } c \in A \text{ and } y_g \in A^* \\ (2, \ell(t_g)) & \text{if } c \in A \text{ and } y_g \notin A^* \\ (3, \text{dcl}_Q(e_{q(g),q(c)}^Q)) & \text{if } c \in \widehat{B} \end{cases}$$

for all $g \in G$ and $c \in C$. To prove property (F2r), it now suffices to show that $e' <_{\Phi} e$ implies $\psi(e') <_{\mathbb{N}^2} \psi(e)$ (where $<_{\mathbb{N}^2}$ is the lexicographic ordering) whenever e', e do not lie in \mathcal{T} and e' is a directed edge on the path $\Phi(e)$.

To that end, let $e = e_{g,c}$ be any directed edge in Γ that does not lie in the tree \mathcal{T} .

Case 1: Suppose that $c \in A$ and $y_g \in A^$.*

Each edge e' in the path $\Phi(e) = \text{path}(y_g, \phi_K(y_g, c))$ has the form $e' = e_{g',c'}$ with $y_{g'} \in A^*$ and $c' \in A$. Moreover, the edge $e_{g',c'}$ lies in the path $\Phi_K(e_{g,c}^K)$, and so $e_{g',c'}^K <_{\Phi_K} e_{g,c}^K$ and $\text{dcl}_K(e_{g',c'}^K) \leq \text{dcl}_K(e_{g,c}^K) - 1$. Now $\psi(e') = (1, \text{dcl}_K(e_{g',c'}^K)) <_{\mathbb{N}^2} (1, \text{dcl}_K(e_{g,c}^K)) = \psi(e)$.

Case 2: Suppose that $c \in A$ and $y_g \notin A^$.*

The first edge in the path $\Phi(e) = \text{path}(y_g, \text{last}(y_g)^{-1} u_{\text{last}(y_g)\text{clast}(y_g)^{-1}} \text{last}(y_g))$ is $e_{y_g, \text{last}(y_g)^{-1}}$; since this is also the last edge of $\text{path}(1, y_g)$, this first edge lies in the tree \mathcal{T} . Similarly, the last edge in $\Phi(e)$ is $e_{g \text{last}(y_g)^{-1} u_{\text{last}(y_g)\text{clast}(y_g)^{-1}, \text{last}(y_g)}}$, and the terminal vertex of this edge is $gc =_G [u_g t_g c t_g^{-1}] t_g = k t_g$ where $k = u_g t_g c t_g^{-1}$ lies in K . Then the normal form of gc can be written as $u_k t_g$, ending in the letter $\text{last}(y_g)$, and so this last edge also lies in \mathcal{T} .

Now suppose that e' is any edge in the path $\Phi(e)$ that does not lie in \mathcal{T} . Then $e' = e_{g',c'}$ satisfies $c' \in A$ and $g' \in G$ has a normal form $u_{g'} t_{g'}$ with $t_{g'} = t_{g'} \text{last}(y_g)$. In this case either $t_{g'} = \lambda$ and $\psi(e') = (1, \text{dcl}_K(e_{g',c'}^K))$, or $t_{g'} \neq \lambda$ and $\psi(e') = (2, l(t_{g'})) = (2, l(t_g) - 1)$.

In either subcase, $\psi(e') <_{\mathbb{N}^2} (2, l(t_g)) = \psi(e)$.

Case 3: Suppose that $c \in \widehat{B}$.

In this last case the path $\Phi(e) = \text{path}(y_g, u_{c(\text{hat}(\phi_Q(q(g), q(c))))^{-1}} \text{hat}(\phi_Q(q(g), q(c))))$ is the concatenation of two paths $\rho_1 = \text{path}(y_g, u_{c(\text{hat}(\phi_Q(q(g), q(c))))^{-1}})$ and $\rho_2 = \text{path}(y_{g'}, \text{hat}(\phi_Q(q(g), q(c))))$, where $g' := g u_{c(\text{hat}(\phi_Q(q(g), q(c))))^{-1}}$ is the vertex at the terminus of ρ_1 and the start of ρ_2 , and we have $\psi(e) = (3, \text{dcl}_Q(e_{q(g), q(c)}^Q))$.

For any edge e' in the first subpath ρ_1 , the label on the edge e' is an element of A , and so $\psi(e') = (m, n)$ with $m < 3$. Hence $\psi(e') <_{\mathbb{N}^2} \psi(e)$.

To analyze the situation for an edge e' of the path ρ_2 , we first note that the initial vertex g' of ρ_2 satisfies $q(g') = q(g)$. Then e' has the form $e' = e_{u' \text{hat}(t'), \widehat{c}}$ for some $u' \in \mathcal{N}_K$ and some edge $e_{t', c'}^Q$ of the path $\Phi_Q(e_{q(g), q(c)}^Q)$. Then $\text{dcl}_Q(e_{t', c'}^Q) < \text{dcl}_Q(e_{q(g), q(c)}^Q)$, and $\psi(e') = (3, \text{dcl}_Q(e_{q(u' \text{hat}(t')), q(\widehat{c})}^Q)) = (3, \text{dcl}_Q(e_{t', c'}^Q)) <_{\mathbb{N}^2} (3, \text{dcl}_Q(e_{q(g), q(c)}^Q)) = \psi(e)$.

This completes the proof of property (F2r) for Φ , and so Φ is a flow function. Let k_K and k_Q be the bounds on the flow functions Φ_K and Φ_Q . Let $M := \max\{l(u_{dcd^{-1}}) \mid d \in \widehat{B} \text{ and } c \in A\}$, and $m := \max\{l(u_{cz}) \mid c \in \widehat{B} \text{ and } z \text{ is in the (finite) image of } \phi_Q\}$. Then $\max\{k_K, 2 + M, k_Q + m\}$ is a bound for the flow function Φ .

Step 2: Autostackable.

In this step we assume that the groups K and Q are autostackable, and in particular that the sets $\text{graph}(\phi_K)$ and $\text{graph}(\phi_Q)$ are synchronously regular. We partition the finite image sets $\text{im}(\phi_K) \subset A^*$ and $\text{im}(\phi_Q) \subset B^*$ as follows. For each $c \in A$, let $U_c := \{\phi(y, c) \mid y \in \mathcal{N}_K\}$, and for each $c \in \widehat{B}$, let $V_c := \{\phi(y, \widehat{c}) \mid y \in \mathcal{N}_Q\}$; that is, U_c is the finite set of labels on paths obtained from the flow function ϕ_K action on edges with label c , and similarly for V_c .

The stacking function associated to the bounded flow function Φ for G from Step 1 of this proof is the function ϕ defined in Step 1. Using the piecewise definition of ϕ , we have

$$\begin{aligned} \text{graph}(\phi) = & (\cup_{c \in A, z \in U_c} L_{c,z} \times \{c\} \times \{z\}) \\ & \cup (\cup_{c \in A, b \in \widehat{B}} L'_{c,b} \times \{c\} \times \{b^{-1}u_{bcb^{-1}}b\}) \\ & \cup (\cup_{c \in \widehat{B}, z \in V_c} L''_{c,z} \times \{c\} \times \{u_{c(\text{hat}(z))^{-1}}\text{hat}(z)\}), \end{aligned}$$

where

$$\begin{aligned} L_{c,z} &= \{y_g \in \mathcal{N}_G \mid y_g \in A^* \text{ and } \phi_K(y_g, c) = z\}, \\ L'_{c,b} &= \{y_g \in \mathcal{N}_G \mid y_g \notin A^* \text{ and } \text{last}(y_g) = b\}, \text{ and} \\ L''_{c,z} &= \{y_g \in \mathcal{N}_G \mid \phi_Q(q(y_g), q(c)) = z\}. \end{aligned}$$

The first language $L_{c,z}$ is the set $\text{proj}_1(\text{graph}(\phi_K) \cap (A^* \times \{c\} \times \{z\}))$. Synchronous regularity of both sets in the intersection, along with Lemma 2.3, shows that $L_{c,z}$ is regular. The second language is $L'_{c,b} = \mathcal{N}_G \cap C^*b$. Now $\mathcal{N}_K = \text{proj}_1(\text{graph}(\phi_K))$ and $\text{hat}(\mathcal{N}_Q)$ is a homomorphic image (via the map hat) of $\text{proj}_1(\text{graph}(\phi_Q))$, so these languages, as well as their concatenation \mathcal{N}_G , are regular, and therefore so is $L'_{c,b}$. Finally, $L''_{c,z}$ is the concatenation $L''_{c,z} = \mathcal{N}_K \text{hat}(\text{proj}_1(\text{graph}(\phi_Q) \cap (B^* \times \{q(c)\} \times \{z\})))$; similar arguments show that $L''_{c,z}$ is also regular. Using the closure of synchronously regular languages under finite unions and Lemma 2.2 now shows that $\text{graph}(\phi)$ is also regular. Thus G is autostackable over C .

Step 3: Algorithmically stackable.

Again the proof in this step is nearly identical to the proof of Step 2. \square

3.3. Finite index supergroups.

In this section we show that a group containing a stackable, algorithmically stackable or autostackable finite index subgroup must also have the same property. While there are many similarities with the result and proof in Section 3.2, and so we do not include all of the details of the proof, the argument in the present section requires a different flow function because we do not require the subgroup to be normal.

Theorem 3.4. *Let H be an autostackable [respectively, stackable, algorithmically stackable] group on a finite inverse-closed generating set A , let G be a group containing H as a subgroup of finite index, and let $S \subseteq G$ be a set of coset representatives for G/H containing 1. Then the group G with the generating set $A \cup (S \setminus \{1\})^{\pm 1}$ is also autostackable [respectively, stackable, algorithmically stackable].*

Proof. Let \mathcal{N}_H , Φ_H , and ϕ_H be the normal form set, bounded flow function, and associated stacking map for H over A , and let $H = \langle A \mid R \rangle$ be the finite presentation obtained from this flow function. Let $B := (S \setminus \{1\})^{\pm 1}$ and let $C := A \cup B$.

Since S is a transversal, the set

$$\mathcal{N}_G := \mathcal{N}_H \cup \{ut \mid u \in \mathcal{N}_H, t \in S \setminus \{1\}\}$$

is a set of normal forms for G over C . Moreover, prefix-closure of the language \mathcal{N}_H implies that \mathcal{N}_G is also prefix-closed.

Let Γ be the Cayley graph for G with respect to C , with sets \vec{E} and \vec{P} of directed edges and paths, and let \mathcal{T} be the maximal tree in Γ determined by the set \mathcal{N}_G of normal forms.

Step 1: Stackable.

Given any $g \in G$, we write the normal form from \mathcal{N}_G for g as $y_g = u_g t_g$ where $u_g \in \mathcal{N}_H$ and $t_g \in \{\lambda\} \cup (S \setminus \{1\})$. In this step we build a flow function whose associated presentation is the finite presentation

$$G = \langle C \mid R \cup \{x = u_x t_x \mid x \in B \setminus S\} \cup \{xy = u_{xy} t_{xy} \mid x \in B, y \in C\} \rangle$$

of the group G .

Define a function $\phi : \mathcal{N}_G \times C \rightarrow C^*$ by

$$\phi(y_g, c) = \begin{cases} \phi_H(y_g, c) & \text{if } c \in A \text{ and } y_g \in A^* \\ y_c & \text{if } c \in B \text{ and } y_g \in A^* \\ \text{last}(y_g)^{-1} y_{\text{last}(y_g)c} & \text{if } c \in C \text{ and } y_g \notin A^* \end{cases}$$

Also as usual define $\Phi : \vec{E} \rightarrow \vec{P}$ by $\Phi(e_{g,c}) := \text{path}(y_g, \phi(y_g, c))$.

It is again immediate from the definition of Φ that property (F1) of the definition of flow function holds. To check property (F2d), suppose that $e = e_{g,c}$ is any edge in the tree \mathcal{T} . Then either $y_g c$ or $y_g c^{-1}$ lies in \mathcal{N}_G . If $c \in A$, then $y_g \in A^*$ and either $y_g c$ or $y_g c^{-1}$ is in \mathcal{N}_H . Then property (F2d) for ϕ_H implies that $\phi_G(y_g, c) = \phi_H(y_g, c) = c$. If $c \in B$ and $y_g \in A^*$, then $y_g \neq y_g c^{-1}$, and consequently $y_g c \in \mathcal{N}_G$. Then $c \in S \setminus \{1\}$ and $\phi_G(y_g, c) = y_c = c$. Finally, if $c \in B$ and $y_g \notin A^*$, then $y_g c \notin \mathcal{N}_G$, and so $y_g c^{-1} \in \mathcal{N}_G$. In this case $\text{last}(y_g) = t_g = c^{-1} \in S \setminus \{1\}$, and $\phi_G(y_g, c) = \text{last}(y_g)^{-1} y_{\text{last}(y_g)c} = c y_1 = c$. Then in all cases we have $\Phi(e) = e$; therefore (F2d) holds for Φ .

Next define $\psi : \vec{E} \rightarrow \mathbb{N}^2$ by

$$\psi(e_{g,c}) = \begin{cases} (0, \text{dcl}_H(e_{g,c}^H)) & \text{if } c \in A \text{ and } y_g \in A^* \\ (1, 0) & \text{if } c \in B \text{ and } y_g \in A^* \\ (1, 1) & \text{if } c \in C \text{ and } y_g \notin A^* \end{cases}$$

where (as in the proof of Theorem 3.3), for an edge $e_{g,c}^H$ in the Cayley graph of H over A , $\text{dcl}_H(e_{g,c}^H) \in \mathbb{N}$ is the maximal length of a descending chain $e_{g,c}^H >_{\Phi_H} e' >_{\Phi_H} e'' \dots$. As usual, let $<_{\mathbb{N}^2}$ be the lexicographic order on \mathbb{N}^2 .

Let $e = e_{g,c}$ be any directed edge in Γ whose underlying undirected edge is not in \mathcal{T} .

Case 1: Suppose that $c \in A$ and $y_g \in A^$.*

The proof in this case is similar to Case 1 of Step 1 in the proof of Theorem 3.3.

Case 2: Suppose that $c \in B$ and $y_g \in A^$.*

In this case $\psi(e) = (1, 0)$, and the path $\Phi(e)$ is labeled by the word y_c . Since the edge e is not in \mathcal{T} , the word $y_g c \notin \mathcal{N}_G$, and so $c \in B \setminus S$ and $y_c = u_c t_c$ for some $u_c \in \mathcal{N}_H$ and $t_c \in S \setminus \{1\}$. Each edge e' in the subpath $\text{path}(y_g, u_c)$ of $\Phi(e)$ satisfies $\psi(e') = (0, n)$ for some $n \in \mathbb{N}$, and so $\psi(e') <_{\mathbb{N}^2} \psi(e)$. The final edge e_{gu_c, t_c} of $\Phi(e)$ lies in the tree \mathcal{T} .

Case 3: Suppose that $c \in C$ and $y_g \notin A^$.*

In this case $\psi(e) = (1, 1)$, and the path $\Phi(e)$ is labeled by the word $\text{last}(y_g)^{-1}y_{\text{last}(y_g)c}$. The first edge $e_{g, \text{last}(y_g)^{-1}}$ lies in \mathcal{T} . Since $\text{last}(y_g) = t_g$, the next subpath of $\Phi(e)$ is $\text{path}(u_g, u_{t_g c})$. Any edge e' in this subpath satisfies $\psi(e') = (0, n)$ for some $n \in \mathbb{N}$; hence $\psi(e') <_{\mathbb{N}^2} \psi(e)$. The remainder of the path $\Phi(e)$ is the edge $e_{u_g u_{t_g c}, t_{t_g c}}$, which lies in the tree \mathcal{T} .

We now have that $\psi(e') <_{\mathbb{N}^2} \psi(e)$ whenever $e' <_{\Phi} e$, and so property (F2r) holds and Φ is a flow function. Finally, since A, B, C , and the image $\text{im}(\phi_H)$ of the stacking map for H are finite sets, the flow function Φ is bounded. Therefore G is stackable over C .

Step 2: Autostackable and algorithmically stackable.

The map ϕ from Step 1 is the stacking map for the flow function Φ for G , and the graph of this function can be decomposed as a finite union of sets

$$\begin{aligned} \text{graph}(\phi) = & \left(\bigcup_{c \in A, z \in \{\phi_H(y, c) \mid y \in \mathcal{N}_H\}} \text{proj}_1(\text{graph}(\phi_H) \cap (A^* \times \{c\} \times \{z\})) \right) \times \{c\} \times \{z\} \\ & \bigcup \left(\bigcup_{c \in B} \mathcal{N}_H \times \{c\} \times \{y_c\} \right) \\ & \bigcup \left(\bigcup_{c \in C, s \in S \setminus \{1\}} \mathcal{N}_H s \times \{c\} \times \{s^{-1}y_{sc}\} \right). \end{aligned}$$

With the added assumption that $\text{graph}(\phi_H)$ is either synchronously regular or computable, then $\text{graph}(\phi)$ satisfies the same property. \square

4. HOMOLOGICAL FINITENESS

The purpose of this section is to investigate the homological properties of autostackable groups by studying Stallings' [25] non- FP_3 group

$$G := \langle a, b, c, d, s \mid [a, c] = [a, d] = [b, c] = [b, d] = 1, [s, ab^{-1}] = [s, ac^{-1}] = [s, ad^{-1}] = 1 \rangle$$

with respect to the generating set $A := \{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, s^{\pm 1}\}$.

The group G is an HNN extension, with stable letter s , of the direct product of two free groups of rank 2,

$$H = F_2 \times F_2 = \langle a, b, c, d \mid [a, c] = [a, d] = [b, c] = [b, d] = 1 \rangle$$

generated by the subset $Z := \{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}\}$. Since the relations of this presentation have zero exponent sum as words over Z , given any element $h \in H$, there is a unique number $\text{expsum}(h)$ such that every word over Z representing h has exponent sum $\text{expsum}(h)$. Let N denote the subgroup of H of elements of zero exponent sum. Then N is a normal subgroup (as conjugation preserves exponent sum) of H , and N is generated by ab^{-1} , ac^{-1} , and ad^{-1} . In the HNN extension G , conjugation by the stable letter s is the identity map on N . (See for example [10] for more details.)

The set $\{a^i \mid i \in \mathbb{Z}\}$ is both a left transversal and right transversal of N in H . Let

$$\mathcal{N}_H := \{uv \mid u \in \{a^{\pm 1}, b^{\pm 1}\}^* \text{ and } v \in \{c^{\pm 1}, d^{\pm 1}\}^* \text{ are freely reduced}\};$$

this is a set of normal forms for H over Z . Then

$$\begin{aligned} \mathcal{N}_G := \{ & ws^{\epsilon_1} a^{i_1} s^{\epsilon_2} a^{i_2} \cdots s^{\epsilon_n} a^{i_n} \mid w \in \mathcal{N}_H, n \geq 0, \epsilon_j \in \{\pm 1\} \text{ and } i_j \in \mathbb{Z} \text{ for all } j, \\ & \text{and whenever } i_j = 0, \text{ then } \epsilon_j = \epsilon_{j+1}\} \end{aligned}$$

is a set of normal forms for G over A (using normal forms for HNN extensions; for example, see [22, Theorem IV.2.1]). In fact, the set \mathcal{N}_G is the set of irreducible words of the (infinite) complete rewriting system

$$\begin{aligned} R_G = & \{xx^{-1} \rightarrow \lambda \mid x \in A\} \cup \{yx \rightarrow xy \mid x \in \{a^{\pm 1}, b^{\pm 1}\}, y \in \{c^{\pm 1}, d^{\pm 1}\}\} \\ & \cup \{s^\epsilon a^i y^\eta \rightarrow a^{-\eta} y^\eta s^\epsilon a^{\eta+i} \mid \epsilon, \eta \in \{\pm 1\}, i \in \mathbb{Z}, y \in \{c, d\}\} \\ & \cup \{s^\epsilon a^i b^\eta \rightarrow a^i b^\eta a^{-\eta-i} s^\epsilon a^{\eta+i} \mid \epsilon, \eta \in \{\pm 1\}, i \in \mathbb{Z}\} \end{aligned}$$

for G over A . We also note that the language \mathcal{N}_G is prefix-closed, and so determines a maximal tree in the Cayley graph for G over A .

Theorem 4.1. *Stallings' non- FP_3 group G is autostackable.*

Proof. Let \mathcal{N}_G be the normal form set for G over the generating set A described above, and denote the normal form for any element $g \in G$ by y_g . Let Γ be the Cayley graph of G over A , with sets \vec{E} and \vec{P} of directed edges and paths, respectively, and let \mathcal{T} be the tree in Γ corresponding to the set \mathcal{N}_G of normal forms.

Step 1: Stackable.

Define a function $\phi : \mathcal{N}_G \times A \rightarrow A^*$ by

$$\phi(y_g, x) := \begin{cases} x & \text{if either } y_g x \in \mathcal{N}_G \text{ or } y_g x x^{-1} \in \mathcal{N}_G \\ \text{last}(y_g)^{-1} x \text{last}(y_g) & \text{if } x \in \{a^{\pm 1}, b^{\pm 1}\}, y_g \in Z^*, \text{ and } \text{last}(y_g) \in \{c^{\pm 1}, d^{\pm 1}\} \\ \text{last}(y_g)^{-1} x \text{last}(y_g) & \text{if } x \in \{c^{\pm 1}, d^{\pm 1}\}, y_g \notin Z^*, \text{ and } \text{last}(y_g) \in \{a^{\pm 1}\} \\ c^{-\eta} x c^\eta & \text{if } x \in \{b^{\pm 1}\}, y_g \notin Z^*, \eta \in \{\pm 1\}, \text{ and } \text{last}(y_g) = a^\eta \\ \text{last}(y_g)^{-1} x a^{-\eta} \text{last}(y_g) a^\eta & \text{if } x \in \{b^\eta, c^\eta, d^\eta\} \text{ with } \eta \in \{\pm 1\}, \text{ and } \text{last}(y_g) \in \{s^{\pm 1}\} \end{cases}$$

for all $y_g \in \mathcal{N}_G$ and $x \in A$. In all of the cases that do not appear explicitly, namely when either $[x \in \{s^{\pm 1}\}]$; $[x \in \{a^{\pm 1}, b^{\pm 1}\}, y_g \in Z^*, \text{ and } \text{last}(y_g) \in \{1, a^{\pm 1}, b^{\pm 1}\}]$; $[x \in \{c^{\pm 1}, d^{\pm 1}\}$ and $y_g \in Z^*]$; or $[x \in \{a^{\pm 1}\}$ and $y_g \notin Z^*]$, it follows from the definition of \mathcal{N}_G that either $y_g x$ or $y_g x x^{-1}$ lies in \mathcal{N}_G . Moreover, one can also check that the five cases in the definition of ϕ are disjoint; that is, the function ϕ is well-defined.

Let $\Phi : \vec{E} \rightarrow \vec{P}$ denote the function $\Phi(e_{g,a}) := \text{path}(y_g, \phi(y_g, a))$. It follows immediately from the definition of Φ and the presentation of G that properties (F1) and (F2d) of the definition of flow function hold for Φ .

In order to prove that property (F2r) holds for Φ , we utilize the following function $\psi : \vec{E} \rightarrow \mathbb{N}^3$. Define $\psi(e_{g,x}) := (0, 0, 0)$ if $e_{g,x}$ lies in the tree \mathcal{T} , and if $e_{g,x}$ does not lie in \mathcal{T} , let

$$\psi(e_{g,x}) := \begin{cases} (0, 0, l(\text{sup}_{\{c^{\pm 1}, d^{\pm 1}\}}(y_g))) & \text{if } x \in \{a^{\pm 1}, b^{\pm 1}\} \text{ and } y_g \in Z^* \\ (n_s(y_g), l(\text{sup}_{\{a^{\pm 1}\}}(y_g)), 0) & \text{if } x \in \{c^{\pm 1}, d^{\pm 1}\} \text{ and } y_g \notin Z^* \\ (n_s(y_g), l(\text{sup}_{\{a^{\pm 1}\}}(y_g)), 1) & \text{if } x \in \{b^{\pm 1}\} \text{ and } y_g \notin Z^*, \end{cases}$$

where $n_s(y_g)$ denotes the number of occurrences of $s^{\pm 1}$ in the word y_g . Let $<_{\mathbb{N}^3}$ denote the lexicographical ordering on \mathbb{N}^3 obtained from the standard ordering on \mathbb{N} , a well-founded strict partial ordering. To prove (F2r), then, it suffices to show that whenever $e' <_\Phi e$, then $\psi(e') <_{\mathbb{N}^3} \psi(e)$.

Let $e = e_{g,x} \in \vec{E}$ be any edge whose underlying undirected edge does not lie in \mathcal{T} .

Case 1: Suppose that $x \in \{a^{\pm 1}, b^{\pm 1}\}$, $y_g \in Z^$, and $\text{last}(y_g) \in \{c^{\pm 1}, d^{\pm 1}\}$.*

In this case $\psi(e) = (0, 0, l(\text{suf}_{\{c^{\pm 1}, d^{\pm 1}\}}(y_g)))$. The path $\Phi(e)$ contains three directed edges: $e_1 := e_{g, \text{last}(y_g)^{-1}}$, $e_2 := e_{g \text{last}(y_g)^{-1}, x}$ and $e_3 := e_{g \text{last}(y_g)^{-1} x, \text{last}(y_g)}$. The edge e_1 lies in the tree \mathcal{T} , as $y_g \text{last}(y_g)^{-1} \text{last}(y_g) = y_g$. The edge e_2 either lies in \mathcal{T} , or else its image under ψ is $\psi(e_2) = (0, 0, l(\text{suf}_{\{c^{\pm 1}, d^{\pm 1}\}}(y_g \text{last}(y_g)^{-1})))$. Since $y_g \text{last}(y_g)^{-1}$ is the prefix of y_g consisting of all but the last letter $\text{last}(y_g)$ (which lies in $\{c^{\pm 1}, d^{\pm 1}\}$), we have $l(\text{suf}_{\{c^{\pm 1}, d^{\pm 1}\}}(y_g \text{last}(y_g)^{-1})) = l(\text{suf}_{\{c^{\pm 1}, d^{\pm 1}\}}(y_g)) - 1$ and $\psi(e_2) <_{\mathbb{N}^3} \psi(e)$. To analyze the edge e_3 , we decompose $y_g = uv \text{last}(y_g)$ where $u \in \{a^{\pm 1}, b^{\pm 1}\}^*$ and $v \text{last}(y_g) \in \{c^{\pm 1}, d^{\pm 1}\}^+$ are reduced words. Now the normal forms satisfy $y_g \text{last}(y_g)^{-1} x \text{last}(y_g) = y_g \text{last}(y_g)^{-1} x \text{last}(y_g)$, and so the edge e_3 also lies in the tree \mathcal{T} .

Case 2: Suppose that $x \in \{c^{\pm 1}, d^{\pm 1}\}$, $y_g \notin Z^$, and $\text{last}(y_g) \in \{a^{\pm 1}\}$.*

In this case we have $\psi(e) = (n_s(y_g), l(\text{suf}_{\{a^{\pm 1}\}}(y_g)), 0)$, and there are three directed edges in the path $\Phi(e)$. Similar to case 1, the first of these edges, $e_{g, \text{last}(y_g)^{-1}}$, lies in the tree \mathcal{T} . The second edge, $e_2 := e_{g \text{last}(y_g)^{-1}, x}$ has ψ function value of $\psi(e_2) = (n_s(y_g), l(\text{suf}_{\{a^{\pm 1}\}}(y_g)) - 1, 0) <_{\mathbb{N}^3} \psi(e)$. The third edge is $e_3 := e_{g \text{last}(y_g)^{-1} x, \text{last}(y_g)}$. Applying the rewriting system R_G above shows that the normal form of the word $y_g \text{last}(y_g)^{-1} x$ again contains $n_s(y_g) > 0$ appearances of $s^{\pm 1}$, and therefore the edge e_3 also lies in \mathcal{T} .

Case 3: Suppose that $x \in \{b^{\pm 1}\}$, $y_g \notin Z^$, $\eta \in \{\pm 1\}$, and $\text{last}(y_g) = a^\eta$.*

Now $\psi(e) = (n_s(y_g), l(\text{suf}_{\{a^{\pm 1}\}}(y_g)), 1)$, and the path $\Phi(e)$ contains three directed edges: $e_1 := e_{g, c^{-\eta}}$, $e_2 := e_{g c^{-\eta}, x}$ and $e_3 := e_{g c^{-\eta} x, c^\eta}$. Unlike the previous cases, none of these edges lie in \mathcal{T} . The edge e_1 satisfies $\psi(e_1) = (n_s(y_g), l(\text{suf}_{\{a^{\pm 1}\}}(y_g)), 0) <_{\mathbb{N}^3} \psi(e)$.

For the analysis of the other two edges, we first use the definition of the set \mathcal{N}_G to write out the normal form $y_g = w s^{\epsilon_1} a^{i_1} \dots s^{\epsilon_n} a^{i_n}$ where $w \in \mathcal{N}_H$, $n > 0$, $\epsilon_j \in \{\pm 1\}$ and $i_j \in \mathbb{Z}$ for all j , and $i_n/|i_n| = \eta$. Note that with this notation, $\psi(e) = (n, |i_n|, 1)$.

The normal form for $g c^{-\eta}$ is $y_g c^{-\eta} = y_w c^{-\eta} a^\eta s^{\epsilon_1} a^{i_1} \dots s^{\epsilon_n} a^{i_n - \eta}$. Therefore the image of e_2 under ψ satisfies $\psi(e_2) = (n, |i_n| - 1, 1) <_{\mathbb{N}^3} \psi(e)$.

Writing $x = b^\beta$ with $\beta \in \{\pm 1\}$, then the value of $\psi(e_3)$ depends upon the sign of the product $\beta \cdot \eta$. If $\beta \cdot \eta = 1$, then the normal form of the element $g c^{-\eta} x$ of G is $y_h s^{\epsilon_1} a^{i_1} \dots s^{\epsilon_n} a^{i_n}$ where $h = {}_H w a^{i_1 + \dots + i_n} c^{-\eta} x a^{-(i_1 + \dots + i_n)}$. In this subcase, then, $\psi(e_3) = (n, |i_n|, 0) <_{\mathbb{N}^3} \psi(e)$. On the other hand if $\beta \cdot \eta = -1$, then the normal form of $g c^{-\eta} x$ is $y_{h'} s^{\epsilon_1} a^{i_1} \dots s^{\epsilon_n} a^{i_n - 2\eta}$, where $h' = {}_H w a^{i_1 + \dots + i_n} c^{-\eta} x a^{-(i_1 + \dots + i_n) + 2\eta}$. Since $\eta = i_n/|i_n|$, then $|i_n - 2\eta| \leq |i_n|$. Therefore in this subcase we have $\psi(e_3) = (n, |i_n - 2\eta|, 0) <_{\mathbb{N}^3} \psi(e)$.

Case 4: Suppose that $x \in \{b^\eta, c^\eta, d^\eta\}$ with $\eta \in \{\pm 1\}$, and $\text{last}(y_g) \in \{s^{\pm 1}\}$.

In this case $\psi(e) = (n_s(y_g), 0, m)$ (with $m \in \{0, 1\}$ depending on x) and $\Phi(e)$ contains five directed edges: $e_1 := e_{g, \text{last}(y_g)^{-1}}$, $e_2 := e_{g \text{last}(y_g)^{-1}, x}$, $e_3 := e_{g \text{last}(y_g)^{-1} x, a^{-\eta}}$, $e_4 := e_{g \text{last}(y_g)^{-1} x a^{-\eta}, \text{last}(g)}$ and $e_5 := e_{g \text{last}(y_g)^{-1} x a^{-\eta} \text{last}(g), a^\eta}$. Edges e_1 and e_4 both lie in the tree \mathcal{T} , since every edge labeled by $s^{\pm 1}$ lies in this tree. The initial vertex of the edge e_5 is the element $g' = {}_G g \text{last}(y_g)^{-1} x a^{-\eta} \text{last}(g)$ of G ; since $g \notin H$ and $\text{last}(y_g)^{-1} x a^{-\eta} \text{last}(g) \in H$, then $g' \notin H$ and $y_{g'} \notin Z^*$. Hence the edge e_5 also lies in \mathcal{T} , as any edge labeled by the letter $a^{\pm 1}$ with initial vertex having a normal form outside of Z^* lies in \mathcal{T} .

If $n_s(y_g) > 1$, then $\psi(e_2) = (n_s(y_g) - 1, l(\text{sup}_{\{a^{\pm 1}\}}(y_g \text{last}(y_g)^{-1})), m)$ and thus $\psi(e_2) <_{\mathbb{N}^3} \psi(e)$. Moreover when $n_s(y_g) > 1$, the argument above showing that e_5 lies in \mathcal{T} applies to show that e_3 lies in \mathcal{T} as well. On the other hand, if $n_s(y_g) = 1$, then the image via ψ for both e_2 and e_3 has the form $(0, 0, l(\text{sup}_{\{c^{\pm 1}, d^{\pm 1}\}}(y)))$ for a word $y \in \mathcal{N}_H$, or else is $(0, 0, 0)$. In this case, we also have both $\psi(e_2) <_{\mathbb{N}^3} \psi(e)$ and $\psi(e_3) <_{\mathbb{N}^3} \psi(e)$.

These four cases show that for any directed edge e' that is in $\Phi(e)$ but not in \mathcal{T} , the inequality $\psi(e') <_{\mathbb{N}^3} \psi(e)$ holds. Hence property (F2r) holds for the function Φ , and Φ is a flow function. Moreover, this flow function is bounded, with bounding constant $k = 5$.

Step 2: Autostackable.

The function ϕ defined in Step 1 of this proof is the stacking function associated to the bounded flow function Φ . It remains for us to show that the language $\text{graph}(\phi)$ is synchronously regular. As in our earlier proofs, we proceed by expressing $\text{graph}(\phi)$ as a union of other languages, using the piecewise definition of ϕ from Step 1:

$$\begin{aligned} \text{graph}(\phi) = & (\cup_{x \in A} L_x \times \{x\} \times \{x\}) \\ & \cup (\cup_{x \in \{a^{\pm 1}, b^{\pm 1}\}, z \in \{c^{\pm 1}, d^{\pm 1}\}} L_{x,z} \times \{x\} \times \{z^{-1}xz\}) \\ & \cup (\cup_{x \in \{c^{\pm 1}, d^{\pm 1}\}, z \in \{a^{\pm 1}\}} L'_{x,z} \times \{x\} \times \{z^{-1}xz\}) \\ & \cup (\cup_{x \in \{b^{\pm 1}\}, \eta \in \{\pm 1\}} L_{x,\eta} \times \{x\} \times \{c^{-\eta}xc^\eta\}) \\ & \cup (\cup_{\eta \in \{\pm 1\}, x \in \{b^\eta, c^\eta, d^\eta\}, z \in \{s^{\pm 1}\}} L_{\eta,x,z} \times \{x\} \times \{z^{-1}xa^{-\eta}za^\eta\}), \end{aligned}$$

where

$$\begin{aligned} L_x &= \{y_g \in \mathcal{N}_G \mid \text{either } y_g x \in \mathcal{N}_G \text{ or } y_g x^{-1} \in \mathcal{N}_G\}, \\ L_{x,z} &= \{y_g \in \mathcal{N}_G \mid y_g \in Z^* \text{ and } \text{last}(y_g) = z\}, \\ L'_{x,z} &= \{y_g \in \mathcal{N}_G \mid y_g \notin Z^* \text{ and } \text{last}(y_g) = z\}, \\ L_{x,\eta} &= \{y_g \in \mathcal{N}_G \mid y_g \notin Z^* \text{ and } \text{last}(y_g) = a^\eta\}, \text{ and} \\ L_{\eta,x,z} &= \{y_g \in \mathcal{N}_G \mid \text{last}(y_g) = z\}. \end{aligned}$$

Using Lemma 2.2 and closure of synchronously regular languages under finite unions, it suffices to show that each of the languages L_x , $L_{x,z}$, $L'_{x,z}$, $L_{x,\eta}$, and $L_{\eta,x,z}$ is regular.

We start by considering the set \mathcal{N}_G . This is the set of irreducible words for the rewriting system R_G , and so can be written as $\mathcal{N}_G = A^* \setminus A^* M A^*$ where

$$M = \{xx^{-1} \mid x \in A\} \cup \{c^{\pm 1}, d^{\pm 1}\} \{a^{\pm 1}, b^{\pm 1}\} \cup s^{\pm 1} (a^* \cup (a^{-1})^*) \{b^{\pm 1}, c^{\pm 1}, d^{\pm 1}\}.$$

Closure of the class of regular languages under finite unions and concatenation shows that M is regular; closure under concatenation and complement then shows that \mathcal{N}_G is regular.

The language L_x can be expressed as $L_x = (\mathcal{N}_G/x) \cup (\mathcal{N}_G \cap A^* x^{-1})$. Applying Lemma 2.1 and regularity of \mathcal{N}_G , then L_x is a regular language.

Note that $L_{x,z} = \mathcal{N}_G \cap Z^* \cap A^* z$, $L'_{x,z} = (\mathcal{N}_G \cap A^* z) \setminus Z^*$, $L_{x,\eta} = (\mathcal{N}_G \cap A^* a^\eta) \setminus Z^*$, and $L_{\eta,x,z} = \mathcal{N}_G \cap A^* z$, and so regularity of these languages also follows from regularity of the normal form set \mathcal{N}_G . \square

Theorem 4.1 yields following.

Corollary 4.2. *There is an autostackable group that does not satisfy the homological finiteness condition FP_3 .*

Remark 4.3. Recall from Section 1 that Stallings' group G cannot have a finite complete rewriting system. Earlier in Section 4 (on p. 16), we gave an infinite complete rewriting system for this group. A consequence of Theorem 4.1 is that G must also admit a synchronously regular bounded prefix-rewriting system over the generating set A . For completeness, we record this system in this remark; the prefix-rewriting system is:

$$\begin{aligned} R_G = & \{zx^{-1}x \rightarrow z \mid x \in A, zx^{-1} \in \mathcal{N}_G\} \\ & \cup \{zyx \rightarrow zxy \mid x \in \{a^{\pm 1}, b^{\pm 1}\}, y \in \{c^{\pm 1}, d^{\pm 1}\}, zy \in \mathcal{N}_G \cap Z^*\} \\ & \cup \{zyx \rightarrow zxy \mid x \in \{c^{\pm 1}, d^{\pm 1}\}, y \in \{a^{\pm 1}\}, zy \in \mathcal{N}_G \setminus Z^*\} \\ & \cup \{za^\eta x \rightarrow za^\eta c^{-\eta} x c^\eta \mid x \in \{b^{\pm 1}\}, \eta \in \{\pm 1\}, za^\eta \in \mathcal{N}_G \setminus Z^*\} \\ & \cup \{zs^\epsilon x^\eta \rightarrow zx^\eta a^{-\eta} s^\epsilon a^\eta \mid x \in \{b, c, d\}, \epsilon, \eta \in \{\pm 1\}, zs^\epsilon \in \mathcal{N}_G\}. \end{aligned}$$

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