A COUNTEREXAMPLE TO THE BERNHARD-JABLAN UNKNOTTING CONJECTURE

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Abstract. We show that there is a knot satisfying the property that for each minimal crossing number diagram of the knot and each single crossing of the diagram, changing the crossing results in a diagram for a knot whose unknotting number is at least that of the original knot, thus giving a counterexample to the Bernhard-Jablan Conjecture.

1. Introduction

The unknotting number $u(K)$ of a knot K is one of the most fundamental measures of the complexity of a knot. It is defined as the minimum number of crossing changes, interspersed with isotopy, required to transform a diagram of K to a diagram of the unknot. A wide array of techniques have been employed over the years to compute unknotting numbers, using nearly every new technique that has been introduced into knot theory (see [5],[13],[14],[17],[18],[20],[23],[24],[25],[26] for a selection). Yet to this day there are still nine 10-crossing knots whose unknotting numbers remain unknown, a testament to the difficulty of determining this basic invariant. See the Knotinfo site [4] for the most up-to-date list of unresolved knots.

It is an elementary fact that for every diagram D of K with n crossings, there is a set of $k \leq n/2$ crossings in the diagram which when all k are reversed yields a diagram D' of the unknot. So every diagram has its own 'unknotting number'. It is also a well-known result that the unknotting number of K can be defined, alternatively, as the minimum, over all diagrams D of the knot K, of the unknotting number of the diagram. That is, one can always arrange things so that, in a minimal unknotting sequence, all isotopies come first. It is natural then to try to turn this alternate formulation into an algorithm to compute the unknotting number of a knot K, by trying to limit the number or types of diagrams of K that need to be considered (either with or without intermediate isotopy).

The most natural initial conjecture, namely that we can limit ourselves to the (finitely many) diagrams of K with minimum crossing number, without isotopy, was disproved by Bleiler [2] and Nakanishi [22], who showed that the unique minimal diagram of the knot 10_8 requires three crossing changes to produce the unknot, while $u(10_8) = 2$. There is a crossing change in the diagram which reduces unknotting number, but an isotopy of the resulting knot is required to create the next (and last) needed crossing change.

The next best thing one could hope for is that one of the minimal diagrams for K admits a crossing change which lowers the unknotting number (as is true for the knot

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10_8). This was the conjecture posited by Bernhard [1] and Jablan [9], and which has come to be known as the Bernhard-Jablan Conjecture. For a knot K, we let

$$\mathsf{BJSet}_K = \{K' \mid K' \text{ is obtained from a minimal crossing number projection of } K \text{ by changing a single crossing}\},$$

and we write $u^s_{BJ}(K)$ to denote the *strong Bernhard-Jablan unknotting number* of K, defined as

$$u^s_{BJ}(K) = 1 + \min\{u(K') \mid K' \in \mathsf{BJSet}_K\}.$$

Conjecture 1.1 (Bernhard-Jablan)**.** *Every knot K possesses a minimal-crossing-number diagram D and a crossing in D, such that changing the crossing results in a diagram D' for a knot K' with $u(K') < u(K)$. Equivalently,*

$$u(K) = u^s_{BJ}(K) \tag{1}$$

for all knots K.

If this conjecture were true, then we would have, in principle, an algorithm to compute $u(K)$: construct the finitely many minimal diagrams for K, change each crossing, and compute the unknotting number of each resulting knot (recursively, using the conjecture). The minimum unknotting number found among the resulting knots is one less (since under crossing change, unknotting number can change by at most one) than the unknotting number of K. Unfortunately, this cannot work in general; that is the main result of this paper.

Theorem 1.2. *There is a knot K whose unknotting number is less than or equal to the unknotting numbers of all of the knots obtained by changing any single crossing in each of the minimal crossing diagrams of K. That is, the Bernhard-Jablan Conjecture is false, in general.*

It is an interesting twist that while we now know that the conjecture is false, we do not (yet) know which knot K fails to satisfy Equation (1), and consequently satisfies $u(K) < u^s_{BJ}(K)$. To make this more precise, we define the *weak Bernhard-Jablan unknotting number* $u^w_{BJ}(K)$ of any knot K by induction on the crossing number, as follows. Let $u^w_{BJ}(\text{unknot}) = 0$, and suppose that we have computed $u^w_{BJ}(K)$ for all knots with crossing number $< n$. Let $S = S_n$ be the set of knots with crossing number n, and for each $K \in S$, let T_K be the set obtained from the set BJSet_K (of knots represented by a diagram that can be obtained from a minimal crossing diagram for K by changing a single crossing) by replacing each knot $K' \in T_K$ with crossing number $< n$ with the number $u^w_{BJ}(K')$ (using the inductive assumption). We make repeated passes through the set S and the T_K sets, as follows. For every knot K in S such that T_K contains the number 0 (that is, BJSet_K contains the unknot), define $u^w_{BJ}(K) = 1$ (since K is not the unknot), remove K from S, and for any knots K' remaining in S, replace every instance of K in $T_{K'}$ by the number 1. For each natural number ℓ starting at 1, incremented by 1 at each repetition, we iterate this process: For all $K \in S$ such that $\ell \in T_K$, define $u^w_{BJ}(K) = \ell + 1$, remove K from S, and for any knots K' remaining in S, replace every instance of K in $T_{K'}$ by the number $\ell + 1$.

In Lemma 2.1, we show that $u^w_{BJ}(K)$ is (well-)defined, that is, the process just described will assign a value to $u^w_{BJ}(K)$, and $u(K) \le u^s_{BJ}(K) \le u^w_{BJ}(K)$ for all knots

K. In Corollary 2.2, we show that the Bernhard-Jablan conjecture holds if and only if

$$u(K) = u^w_{BJ}(K) \tag{2}$$

for all knots K. Our proof of Theorem 1.2 shows that Equation (2) fails for the knot $K13n3370$, and shows that Equation (1) fails for one of four knots. More precisely, in Section 2 we show:

Theorem 1.3. *The knots $K12n288$, $K12n491$, $K12n501$, and $K13n3370$ satisfy $K12n288, K12n491, K12n501 \in \mathsf{BJSet}_{K13n3370}$, and the following hold.*

(a) $u(K13n3370) \leq 2 < 3 = u^w_{BJ}(K13n3370)$.
(b) $u^s_{BJ}(K') = u^w_{BJ}(K') = 2$ for all $K' \in \{K12n288, K12n491, K12n501\}$.
(c) $u^s_{BJ}(K13n3370) = 1 + \min\{u(K') \mid K' \in \{K12n288, K12n491, K12n501\}\}$.
(d) For at least one $K \in \{K12n288, K12n491, K12n501, K13n3370\}$ we have $u(K) < u^s_{BJ}(K)$.

The proof that the unknotting number of the knot $K13n3370$ is less than or equal to 2 is given in Section 2 by direct construction. Part (d) of Theorem 1.3 is a direct corollary of parts (a-c): If Equation (1) holds for the three 12-crossing knots, then by part (b) and Lemma 2.1(2) they all have unknotting number equal to 2, and so by part (c) the strong Bernhard-Jablan unknotting number for $K13n3370$ is 3, and then part (a) shows that this 13-crossing knot fails Equation (1).

The bulk of the work needed to reach these conclusions was carried out by computer. In particular, the diagram which establishes that $u(K13n3370) \leq 2$ was found by a random search, using the program SnapPy [6] to generate and identify knots as well as identify the knots obtained by crossing change. That search found a projection of the knot $K13n3370$ which a single crossing change turned into a diagram for the knot $K11n21$, which has unknotting number one. [To be completely accurate, it found the diagram for $K11n21$ first.] The results on weak and strong Bernhard-Jablan unknotting numbers are established using an exhaustive search to identify all of the minimal crossing diagrams of the knots involved, as an incidental consequence of identifying all minimal crossing diagrams of all knots through 14 crossings. This last computation was also carried out within SnapPy, and later independently verified using the program Knotscape [8]. The code used to carry out these computations can be found on the authors' website, at the URL listed on page 11 below.

In Section 3, we provide information on further examples of knots for which the unknotting and weak Bernhard-Jablan unknotting numbers differ. In Section 4, we discuss several open questions that arise from Theorem 1.2.

All of these computations were in fact part of a larger project, whose goal is to fill in the gaps in our knowledge of the unknotting numbers of low crossing-number knots. By using the fact that unknotting number changes by at most one under crossing change, we can use the knowledge of the unknotting numbers of 'crossing-adjacent' knots to pull a lower bound L for $u(K)$ up (by finding an adjacent knot K' with $u(K') > L + 1$) or pull an upper bound U down (by finding adjacent K' with $u(K') < U - 1$). Pulling lower bounds up is a relatively routine occurrence in our computations (and often results in the determination of the unknotting number); the first instance where an upper bound was pulled down forms the core of this paper.

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2. The proof of Theorem 1.2

2.1. **The three unknotting numbers.** We begin the proof of Theorem 1.2 with two lemmas on the relationships between the three types of unknotting numbers.

Lemma 2.1. *Let K be a knot.*

(1) The weak Bernhard-Jablan unknotting number is well-defined and satisfies

$$u^w_{BJ}(K) = 1 + \min\{u^w_{BJ}(K') \mid K' \in \mathsf{BJSet}_K\}.$$

(2) $u(K) \leq u^s_{BJ}(K) \leq u^w_{BJ}(K)$.

Proof. In order to show that $u^w_{BJ}(K)$ is well-defined, we must show that in the iterative procedure passing through the set S and the sets T_K (for $K \in S$), eventually $S = \emptyset$ and $u^w_{BJ}(K)$ is defined for every knot with crossing number n. Suppose that $K \in S$, let D be any minimal crossing number diagram for K, and let $T_{D,k}$ be the set of knots represented by diagrams obtained by changing any k crossings of the diagram D. By the elementary fact noted above (bounding the number of crossing changes needed in a minimal crossing diagram to produce the unknot), for some $1 \leq k \leq n/2$, the sets $T_{D,0}, ..., T_{D,k-1}$ do not contain a knot with crossing number $< n$, but the set $T_{D,k}$ does contain a knot K_k with crossing number $< n$. Now K_k is represented by a diagram obtained from D by changing k crossings $c_1, ..., c_k$. Moreover, the knots K_i represented by the diagram obtained from D by changing the crossings $c_1, ..., c_i$ satisfy that K_i has crossing number n and $K_{i+1} \in \mathsf{BJSet}_{K_i}$ for all $i < k$, and $K = K_0$. Now the set $T_{K_{k-1}}$ contains the number $u^w_{BJ}(K_k)$, and so in the procedure above, after at most $1+u^w_{BJ}(K_k)$ passes through S and the $T_{\tilde{K}}$ sets, $u^w_{BJ}(K_{k-1})$ is assigned a value $\leq 1 + u^w_{BJ}(K_k)$. Then after at most $1 + u^w_{BJ}(K_{k-1})$ passes, $u^w_{BJ}(K_{k-2})$ is assigned a value $\leq 1 + u^w_{BJ}(K_{k-1})$, etc. Hence after finitely many passes, $u^w_{BJ}(K) = u^w_{BJ}(K_0)$ is assigned a value.

From the definition of $u^w_{BJ}(K)$, at the point in the process in which $u^w_{BJ}(K) = r$ is defined each knot K' in the set BJSet_K is either in T_K or else has been replaced in T_K by the number $u^w_{BJ}(K')$. Moreover, at that point $r-1$ is the least number appearing in T_K, and all of the knots K' in T_K satisfy $u^w_{BJ}(K') \geq r$ as well, since they will be assigned values later in the process, concluding the proof of part (1).

The inequality $u(K) \leq u^s_{BJ}(K)$ is immediate from the definitions, since for every $K' \in \mathsf{BJSet}_K$, we have $|u(K') - u(K)| \leq 1$, so $u(K') \geq u(K) - 1$. An inductive argument shows that $u(K) \leq u^w_{BJ}(K)$, since $u^w_{BJ}(K) = u^w_{BJ}(K') + 1 \geq u(K') + 1 \geq u(K)$ for some $K' \in \mathsf{BJSet}_K$ with $u^w_{BK}(K') < u^w_{BJ}(K)$, and therefore (using part (1)) $u^s_{BJ}(K) \leq u^w_{BJ}(K)$ as well. □

Corollary 2.2. *The Bernhard-Jablan Conjecture holds if and only if $u(K) = u^w_{BJ}(K)$ for all knots K.*

Proof. If $u(K) = u^w_{BJ}(K)$ (Equation (2)) holds for all knots K, then Lemma 2.1(2) shows that Equation (1) holds for all K as well.

Conversely, suppose that the Bernhard-Jablan Conjecture holds. We prove Equation (2) by induction on $u(K)$. Note that $u(K) = 0$ implies that K is the unknot, and so $u^w_{BJ}(K) = 0$ also. Suppose that Equation (2) holds for all knots with unknotting number $< n$, and suppose that $u(K) = n$. By Equation (1) and the definition of u^s_{BJ}, $u(K) = 1 + \min\{u(K') \mid K' \in \mathsf{BJSet}_K\}$, and so by the inductive assumption $u(K) = 1 + \min\{u^w_{BJ}(K') \mid K' \in \mathsf{BJSet}_K\}$, since the smallest $u(K')$ appearing is less than $u(K) = n$, and so equals $u^w_{BJ}(K')$, and for all other $K'' \in \mathsf{BJSet}_K$ we have $u^w_{BJ}(K'') \geq u(K'') \geq u(K') = u^w_{BJ}(K')$. The conclusion follows from Lemma 2.1(1). □

2.2. **Computational resources.** There are three resources that we used to assemble the data needed to establish the results in Theorem 1.3. The first of these is *SnapPy* [6], an indispensible program for studying knots and 3-manifolds, based on the program SnapPea, and developed by Culler, Dunfield, Goerner and Weeks. In particular, SnapPy was used to build knots as (random) braids, and to identify them using SnapPy's *identify()* command. This utility uses a hash based on the knot's geometric properties to compare to a list, to determine the likely identity of the knot, and then uses an underlying canonical triangulation of the knot complement to find a combinatorial isomorphism between ideal triangulations of a given knot complement and a reference knot complement, which provides a homeomorphism of knot complements and so, by the solution to the Knot Complement Problem by Gordon and Luecke [7], identifies the knot. We also detected the unknot using the *fundamental_group()* command, since the unknot is the only knot whose knot group has a presentation with no relators, i.e., the knot group is free. Although it is not relevant to the specific examples that we will discuss, SnapPy's *deconnect_sum()* command was also used to identify non-prime knots and their summands.

The second resource we used is Knotscape [8], developed by Hoste and Thistlethwaite. We used this program to provide an independent verification of the database of minimal crossing diagrams for knots with a fixed crossing number, discussed below; Knotscape works directly with Dowker-Thistlethwaite (DT) codes, and applies combinatorial moves on these codes to match a knot projection with a unique normal form, thereby identifying the knot. It also, incidentally, can be much faster than SnapPy at the task of identifying thousands or millions of knots in a row, since it directly manipulates a knot's DT code.

The third resource we used is *Knotinfo* [4], the online database of knot invariants maintained by Jae Choon Cha and Chuck Livingston at the University of Indiana. This was used to populate lists of knots with known unknotting number, as well as initial upper and lower bounds for knots with unknown unknotting number, in order to make the needed comparisons with 'crossing-adjacent' knots.

Taken together, these tools gave us the foundation on which to build our search process, and assemble the data described below.

2.3. **Computation of BJSet_K.** In this subsection we describe our procedure for constructing the set BJSet_K for any k-crossing knot K. Roughly, this is done by exhaustively identifying every knot associated to every k-crossing diagram up to flypes (Figure 1). We carried out these calculations using SnapPy to identify the knots, and, as an independent check, carried out the same exhaustion using Knotscape [8], again using SnapPy to make the identifications.

More precisely, for a fixed natural number k, we start with a set of DT codes (as used in SnapPy) containing a representative for a minimal diagram of each alternating k-crossing knot; we refer to the associated set of knot diagrams as the set $\mathcal{R}$ of *reference diagrams* for this procedure.

Let $\mathcal{D}$ be the set of all DT codes for diagrams obtained by changing a subset of the crossings of a reference diagram (omitting changing the lexicographically first crossing in each DT code, since including it would only produce the mirror images of the other knots built). SnapPy can then be used to identify the knots associated to the DT codes in $\mathcal{D}$; let $\mathcal{D}_{\mathcal{K}}$ be the subset of $\mathcal{D}$ consisting of the projections of K obtained from the reference diagrams by crossing changes. Finally, again use SnapPy to identify the set $\mathsf{RefBJSet}_K$ of knots represented by diagrams obtained from elements of $\mathcal{D}_K$ by a single crossing change.

In the following, we show that the collection BJSet_K of knots we obtain from all k-crossing projections of a k-crossing knot K by changing a single crossing is the same as the collection $\mathsf{RefBJSet}_K$ we obtain by using the projections of K obtained from crossing changes made to reference diagrams alone; hence the algorithm above computes the set BJSet_K, and the diagrams obtained from the reference diagrams by a single crossing change suffice for the computations needed in the strong and weak Bernhard-Jablan unknotting numbers. This perspective is implicit in some of the work of Jablan and Sazdanovic [10] and of Zeković, Jablan, Kauffman, Sazdanovic and Stošić [29] on this conjecture.

Proposition 2.3. *For any knot K, $\mathsf{BJSet}_K = \mathsf{RefBJSet}_K$.*

Proof. It is immediate that $\mathsf{RefBJSet}_K \subseteq \mathsf{BJSet}_K$.

Let K_1 be any element of BJSet_K. There is a k-crossing projection D of the k-crossing knot K, and a diagram D_1 obtained from D by changing a single crossing, such that D_1 is a projection of K_1. Note that D is reduced (i.e., D has no nugatory crossing) since such a k-crossing diagram with a nugatory crossing cannot represent a k-crossing knot. Build the associated knot shadow (a k-vertex 4-valent graph) by replacing each crossing with a vertex of valence 4, and from this shadow, reintroduce crossings to obtain a reduced diagram D' for an alternating knot. The diagram D' must have crossing number k [12],[21],[28], and by the proof of the Tait Flyping Conjecture [19] the diagram D' is related to the reference diagram R of the same alternating knot by a sequence of flypes (Figure 1).

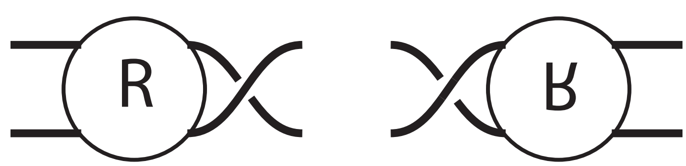

FIGURE 1. Flype

By applying Lemma 2.4 below (with D, $\widetilde{D}$, and D_1 in that Lemma playing the roles of D', R, and D, respectively) a finite number of times, there is a diagram $\widetilde{D}$

obtained by changing crossings in R such that D and $\widetilde{D}$ are related by a sequence of flypes. Then $\widetilde{D}$ is represented by a DT code in the set $\mathcal{D}$. Moreover, since the flype move does not change the knot represented by the diagram, the knots represented by D and $\widetilde{D}$ are the same, and so $\widetilde{D} \in \mathcal{D}_K$ is another minimal diagram for the knot K.

Applying Lemma 2.4 another time, there is a diagram $\widetilde{D}_1$ obtained by changing a single crossing in $\widetilde{D}_1$ such that D_1 and $\widetilde{D}_1$ are related by a sequence of flypes. Then $\widetilde{D}_1$ is another projection of the knot K_1, and since $\widetilde{D}_1$ is obtained by a single crossing change from a diagram represented in $\mathcal{D}_K$, then K_1 is in $\mathsf{RefBJSet}_K$. □

Lemma 2.4. *If knot diagrams D and $\widetilde{D}$ are related by a sequence of flypes, then for any diagram D_1 obtained by changing a crossing in D there is a diagram $\widetilde{D}_1$ obtained by changing a crossing in $\widetilde{D}_1$ such that D_1 and $\widetilde{D}_1$ are related by a sequence of flypes.*

Proof. This can be established by induction on the number of flypes used, by showing that a crossing change followed by a flype yields the same diagram as the same flype followed by some crossing change; therefore several flypes followed by a crossing change is the same as some crossing change followed by several flypes. There are three cases to consider. (1) If the crossing change is outside of the tangle being flyped, then the same crossing change applied after the flype will yield the diagram $\widetilde{D}_1$ related to D_1 by the same flype. (2) If the crossing change is inside of the tangle being flyped, then the crossing is carried to a crossing in the new diagram by the flype, and changing that corresponding crossing after the flyped yields the same diagram as crossing change followed by flype. (3) If the crossing change is the one which is removed by the flype, then applying the crossing change followed by the flype yields the same diagram obtained by first applying the opposite flype and then changing the crossing created by the flype (illustrated in Figure 2).

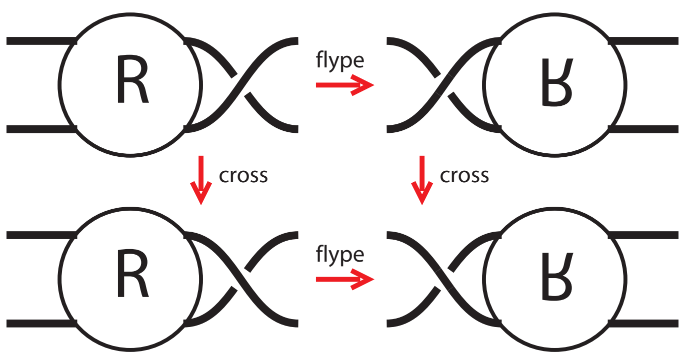

FIGURE 2. Flype and crossing change

□

Thus we lose no information about the knots that we need to consider in computing the weak Bernhard-Jablan unknotting number if we only work with the diagrams $\widetilde{D}$ for K obtained by changing crossings in our reference diagrams.

2.4. **Proof of Theorem 1.3.** The remainder of the proof of Theorem 1.2 in this section is the proof of parts (a)-(c) of Theorem 1.3. We begin with part (a) of that theorem.

Lemma 2.5. *The knot* $K13n3370$ *has unknotting number less than or equal to 2.*

Proof. The knot $K13n3370$ is the closure of the 7-braid

$$\{1, 1, -3, 4, -3, -5, 5, -6, -6, 4, -5, 2, 4, -6, 3, 4, -1, 3, 5, 2\},$$

with 20 crossings, using the notational convention of Knotinfo, shown in Figure 3.

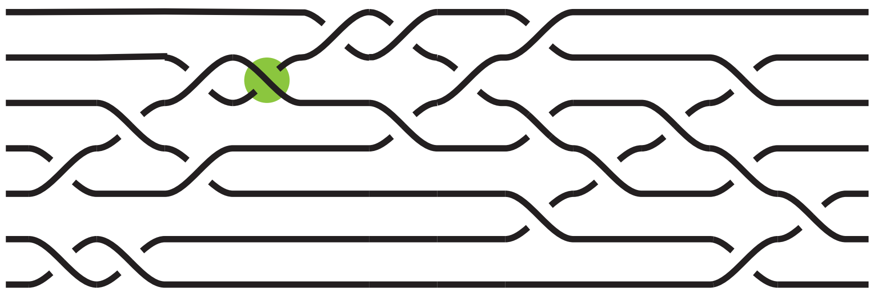

FIGURE 3. $K13n3370$

SnapPy can convert this braid representation to a Dowker-Thistlethwaite (DT) code for $K13n3370$, using the command *DT_code()*, which SnapPy handles more natively than braid notation, giving us the DT code

$$[-14, -10, -22, 20, 34, -36, 40, -24, -8, 6, -18, -32, 2, 38, 26, -12, -16, 4, -30, 28].$$

SnapPy verifies that this code represents the knot $K13n3370$. Changing the seventh crossing in the code gives us the DT code

$$[-14, -10, -22, 20, 34, -36, -40, -24, -8, 6, -18, -32, 2, 38, 26, -12, -16, 4, -30, 28]$$

which SnapPy identifies as the knot $K11n21$. From the Knotinfo database, we find that $u(K11n21) = 1$; in fact, it is the case that all 23 minimal crossing diagrams for the knot $K11n21$ have at least one crossing which yields the unknot when changed. DT codes for these 23 diagrams can be found in an appendix at the end of this paper.

This shows that a single crossing change to the given diagram for $K13n3370$ yields a knot with unknotting number 1; therefore, $K13n3370$ has unknotting number at most 2. □

Next, we show the other result in part (a) of Theorem 1.3, that the weak Bernhard-Jablan unknotting number $u^w_{BJ}(K13n3370)$ is equal to 3, as well as parts (b-c) of that Theorem. In overview, we use the algorithm of Section 2.3 to find the sets $\mathcal{D}_K$ and $\mathsf{RefBJSet}_K$ associated to the knot $K13n3370$, and iteratively find the sets $\mathcal{D}_{K'}$ and $\mathsf{RefBJSet}_{K'}$ for all $K' \in \mathsf{RefBJSet}_K$, etc. We then use the definition of u^w_{BJ} (or iteratively apply Lemma 2.1) and Proposition 2.3, as well as other information on unknotting numbers for knots in the set $\mathsf{BJSet}_{K13n3370}$, to determine the weak Bernhard-Jablan unknotting number of the knot $K13n3370$.

In more detail, working directly with SnapPy (or equivalently, with Knotscape) to build these sets, we find that there are 4878 13-crossing alternating knots, which yield 4878 reference diagrams, and changing every subset of 12 of the crossings in each diagram yields 2^{12} diagrams for each, for a total of $19,980,288$ diagrams for SnapPy to identify. Distributing the work over several machines made this a manageable task. The resulting data can be found at the website described on page 11 of this paper. Among these diagrams there are $64,399$ diagrams which represent 13-crossing knots (also to be found at the website; for comparison, $6,122,841$ of the diagrams represent the unknot), and 24 minimal diagrams for the knot $K13n3370$; their set $\mathcal{D}_{K13n3370}$ of DT codes are listed in the appendix.

Changing each single crossing in each of these 24 diagrams yields 312 diagrams, but only 13 distinct knots, according to SnapPy: they are the knots in the set

$$\mathsf{BJSet}_{K13n3370} = \{7_4, 8_8, 10_{34}, K11a211, K11n91, K11n132, K12a1118, K12n288, K12n333, K12n469, K12n491, K12n501, K12n512\}.$$

For at least 10 of these 13 knots, the unknotting number, and hence (by Lemma 2.1(2)) also the weak Bernhard-Jablan unknotting number, is at least 2.

Lemma 2.6. *The knots 7_4, 8_8, 10_{34}, $K11a211$, $K11n91$, $K11n132$, $K12a1118$, $K12n333$, $K12n469$, and $K12n512$ all have unknotting number at least 2.*

Proof. Lickorish [16] showed that $u(7_4) = 2$, Kanenobu and Murakami [11] showed that $u(8_8) = u(10_{34}) = 2$, and Lewark and McCoy [15], using the smooth 4-ball genus g_4, showed that $u(K11a211) \geq g_4(K11a211) = 2$. Borodzik and Friedl [3], using the algebraic unknotting number u_a, showed that $u(K) \geq u_a(K) = 2$ for $K = K11n91, K11n132, K12a1118, K12n333$, and $K12n469$. The authors showed (see [4]), using the same random search procedure used to discover the projection of $K13n3370$ used in this paper, that the knot $K12n512$ has a projection with a crossing change yielding the knot 9_{38}, and $u(9_{38}) = 3$ was shown by Owens [23], following earlier work of Stoimenow [27]. The relevant braids to see this knot adjacency are

$$\{2, 3, -4, -4, 1, -2, 3, -3, 5, -3, -1, 4, 2, 1, -5, -4, 3, 3, 1, 1, 2\}$$

for $K12n512$ and

$$\{2, 3, -4, 4, 1, -2, 3, -3, 5, -3, -1, 4, 2, 1, -5, -4, 3, 3, 1, 1, 2\}$$

for 9_{38}. Therefore, all of these knots have unknotting number at least 2. □

This leaves $K12n288, K12n491$, and $K12n501$, whose unknotting numbers at the time of this writing are still unknown. But the same exhaustive procedure from Section 2.3 applies to find the set $\mathcal{D}_K$ of minimal crossing diagrams for each $K \in \{K12n288, K12n491, K12n501\}$, as well as, incidentally, the minimal diagrams (up to flypes) of all 12-crossing knots. The data set for all 12-crossing knots is available on the website discussed below (on page 11). These three knots have, respectively, 24, 9, and 18 such diagrams, which are given in the appendix. SnapPy verifies that no single crossing change in any of these 51 diagrams yields the unknot, and so (using Proposition 2.3) the sets BJSet_K for $K \in \{K12n288, K12n491, K12n501\}$ do not contain the unknot. Hence these three 12-crossing knots all have both strong and weak Bernhard-Jablan unknotting number at least 2. Lemma 2.1(1) now shows that $u^w_{BJ}(K13n3370) \geq 3$.

To show part (b) of Theorem 1.3, it now suffices to show the further claim that $u^w_{BJ}(K) \leq 2$ for all $K \in \{K12n288, K12n491, K12n501\}$; we give an explicit construction using SnapPy computations. The knot $K12n288$ has a minimal crossing diagram with DT code

$$[4, \underline{10}, 12, -16, 2, 8, -18, -22, -6, -24, -14, -20];$$

when the second crossing is changed (changing the sign of the underlined entry in the DT code), SnapPy identifies the resulting knot as the trefoil 3_1. The knot $K12n491$ has minimal diagram with DT code

$$[\underline{6}, -12, 20, 18, 24, 16, -4, 22, 8, 2, 14, 10],$$

and changing the first crossing yields the knot 6_3. Finally, the knot $K12n501$ has diagram with DT code

$$[6, -10, \underline{22}, 24, -16, 18, 20, -2, 4, 14, 12, 8],$$

and changing the third crossing results in the knot 8_{13}. It is a straightforward computation to check that every $K' \in \{3_1, 6_3, 8_{13}\}$ has a minimal diagram admitting a single crossing change to the unknot, and hence satisfies $u^w_{BJ}(K') = 1$. (An alternative proof that each $u^w_{BJ}(K') = 1$ follows from the fact that each $u(K') = 1$ (see for example KnotInfo [4]), and a result of McCoy [18] that $u(K') = 1$ implies $u^w_{BJ}(K') = 1$ for all alternating knots.) The claim now follows from Lemma 2.1(1).

Applying Lemma 2.1(1) again, to the knot $K13n3370$, shows the last part of Theorem 1.3(a), that $u^w_{BC}(K13n3370) = 3$, as well as part (c) of Theorem 1.3.

As a consequence of these computations, either one of the knots $K12n288$, $K12n491$, and $K12n501$ has unknotting number 1, and so fails to satisfy Equation (1) in the Bernhard-Jablan Conjecture, or else all of the knots obtained from a minimal diagram of $K13n3370$ by changing a crossing have unknotting number at least 2, and so $K13n3370$ has strong Bernhard-Jablan unknotting number 3 and Equation (1) fails for $K13n3370$. This completes the proof of Theorems 1.3 and 1.2, of the existence of a counterexample to the Bernhard-Jablan Conjecture.

3. Further examples

In further computations we have found more counterexamples to the Bernhard-Jablan Conjecture. The following three pairs $(K, \tilde{K})$ each satisfy that there is a diagram for K with a single crossing change resulting in a diagram for $\tilde{K}$, and $u(\tilde{K}) + 2 \leq u^w_{BJ}(K)$. Since K and $\tilde{K}$ are crossing adjacent, then $u(K) \leq u(K)+1$, and hence $u(K) < u^w_{BJ}(K)$, giving a failure of Equation (2) for the knot K.

Each example indicates a further counterexample to Equation (1) in the Berhhard-Jablan Conjecture either by the knot K, or by a knot in one of the the sets BJSet_K, $\mathsf{BJSet}^2_K = \cup_{K' \in \mathsf{BJSet}_K} \mathsf{BJSet}_{K'}$, or $\mathsf{BJSet}^i_K = \cup_{K' \in \mathsf{BJSet}^{i-1}_K} \mathsf{BJSet}_{K'}$ for some $i \geq 3$.

The first pair is $K = K13n1669$, which has weak Bernhard-Jablan unknotting number 4, and $\tilde{K} = K14n23648$, with unknotting number ≤ 2. Adjacent braid representatives for these two knots are given by:

$$K = K13n1669 : \{-2, -4, -1, -3, -4, -4, -2, -3, 7, -1, -2, 4, -1, 5, 3, -1, -1, -2, 1, 6, 1, 4, 3\}$$
$$\tilde{K} = K14n23648 : \{-2, -4, 1, -3, -4, -4, -2, -3, 7, -1, -2, 4, -1, 5, 3, -1, -1, -2, 1, 6, 1, 4, 3\}$$

The second example is $K = K13n1587$, which has weak Bernhard-Jablan unknotting number 3, and $\tilde{K} = 10_{113}$, with unknotting number 1. Their adjacent braid representations are:

$$K13n1587 : \{-4, -3, -3, -6, 4, 5, -4, 3, -6, 3, -2, 1, 3, 2, 5, 4, 1, 1, 6, -6, -5, 3\}$$
$$10_{113} : \{-4, -3, -3, -6, 4, 5, -4, 3, -6, 3, -2, 1, 3, 2, 5, 4, 1, 1, 6, 6, -5, 3\}$$

Finally, the third pair gives the first example of an alternating knot for which Equation (2) fails. The knots are $K = K14a2539$, which has weak Bernhard-Jablan unknotting number 4, and $\tilde{K} = K14n1045$, with unknotting number ≤ 2. Adjacent braid representations are given by:

$$\begin{aligned} K14a2539 : \{&7, 3, -4, 1, 6, 7, 5, 6, -1, 5, -2, -7, -3, 4, -1, 2, 7, -6, 3, 7,\\ &6, -2, -6, -1, -2, -7, 3, -4, 1, 7, -3, 7, -6, -7, 5, 1, -7, -1, 6\}\\ K14n1045 : \{&7, 3, -4, 1, 6, 7, 5, 6, -1, 5, -2, -7, -3, -4, -1, 2, 7, -6, 3, 7,\\ &6, -2, -6, -1, -2, -7, 3, -4, 1, 7, -3, 7, -6, -7, 5, 1, -7, -1, 6\} \end{aligned}$$

4. The future

With the failure of the Bernhard-Jablan Conjecture, there would seem to be no natural remaining candidate for an algorithmic approach to computing unknotting number. That is, there is no other 'canonical' finite collection of diagrams for a knot in which to posit that an unknotting number minimizer 'should' be found. This could be viewed as a blow to the theory, but we choose to take a different view. For example, despite a massive amount of effort, none of the computable invariants that can be tied to unknotting number (and there are many) have succeeded in showing that any of the nine 10-crossing knots with unknown unknotting number, namely $10_{11}, 10_{47}, 10_{51}, 10_{54}, 10_{61}, 10_{76}, 10_{77}, 10_{79}$ and 10_{100}, have unknotting number 3, although all nine can be shown to have weak Bernhard-Jablan unknotting number equal to 3. With the examples we have found, it may be more reasonable now to suppose that this failure to establish that $u(K) = 3$ for K in this list is in fact because it should not be possible; perhaps some or all of these knots have unknotting number 2, and we just have not yet found the projection(s) to demonstrate this. We are heartened by the fact that the example we found to settle Conjecture 1.1 was provided by a 20-crossing projection (although we should note that the braid that our search program actually found had 37 crossings - it was possible, by hand, to reduce it to the braid word we have provided here), and so it may even be the case that a relatively small projection could provide the knot adjacency needed to show that one or more of these 10-crossing knots have unknotting number 2.

There are almost certainly many more knots out there that provide a counterexample to the Bernhard-Jablan conjecture. All of the code that we have developed as part of this project to uncover unknotting numbers by random search is available at the website

http://www.math.unl.edu/∼mbrittenham2/unknottingsearch/

and we invite anyone who is interested in searching for interesting knot adjacencies to download and run this code. The code to implement random searches in SnapPy is written in python, as is the code for SnapPy to build the databases of minimal crossing projections. SnapPy can natively identify any knot with 14 or fewer crossings, and so this code is implemented to work up through that crossing limit. The code to work with Knotscape provided at the website, which we used to check the database computations, is written in Perl, and makes calls to the Knotscape program 'knotfind' to make its identifications.

Since this code was originally designed to find useful knot adjacencies, which would then identify tighter bounds on unknown unknotting numbers, and there has been no systematic work, other than this project, to compute the unknotting numbers of 13- and 14-crossing knots generally (that we are aware of), further runs of this code are likely to uncover new values of the unknotting numbers of these (as well as possibly smaller) knots. We would be grateful to hear of any such new data that the reader might discover as a consequence of running this code for themselves;

The failure of Conjecture 1.1 in general leads naturally to several still unresolved questions. Is Conjecture 1.1 true for alternating knots? That is, does every minimal alternating projection contain a crossing whose change will lower the unknotting number of the underlying knot? McCoy [18] has shown that this is true for unknotting number 1 alternating knots. (We note that the third example in Section 3 shows an alternating knot for which Equation (2) fails to hold, but does not determine whether Equation (1) holds for that knot.) In Theorem 1.3 we have shown that one of four knots must provide a counterexample to Conjecture 1.1 (and in particular to Equation (1)), but the question remains, which one? What is the smallest (in terms of crossing number) counterexample to the Bernhard-Jablan Conjecture? Are there infinitely many counterexamples?

5. Appendix

To make this paper as independent of external data sources as possible, we list, for reference, the DT codes of all of the minimal crossing diagrams used to carry out the computations above. This will allow the reader to verify for themselves that crossing changes on these diagrams have the properties claimed in the paper. We do not provide the code here that was used to produce these lists, or to show that they are exhaustive; that would make this paper prohibitively long. This code is available at the website listed above.

DT codes for the minimal diagrams, up to flype, of the knot $K11n21$:

$[4, 8, -12, 2, 16, -6, 20, 18, 10, 22, 14]$
$[4, 8, 12, 2, -14, -18, 6, -20, -10, -22, -16]$
$[4, 8, 12, 2, -16, 18, 6, -20, 22, -14, -10]$
$[4, 8, 12, 2, -20, 18, 6, -10, 22, -14, 16]$
$[4, 8, 12, 2, 14, -18, 6, -20, -10, -22, -16]$
$[4, 8, 12, 2, 16, 18, 6, -20, 22, -14, 10]$
$[4, 8, 12, 2, 18, -16, 6, 20, -22, 14, -10]$
$[4, 8, 12, 2, 18, -20, 6, -10, -22, 14, -16]$
$[4, 8, 12, 2, 18, 14, 6, -20, -22, 10, -16]$
$[4, 8, 12, 2, 18, 20, 6, 10, -22, 14, -16]$
$[4, 10, -14, -20, 2, 16, -18, 8, -22, -6, -12]$
$[4, 10, -14, -20, 2, 16, -18, 8, 22, -6, 12]$
$[4, 10, -14, -20, 2, 16, -22, 8, 12, -6, -18]$
$[4, 10, -14, -20, 2, 16, -8, 22, 12, -6, -18]$
$[4, 10, -14, -20, 2, 22, -18, 8, -12, -6, 16]$
$[4, 10, -14, 18, 2, 16, -6, 22, 20, 8, 12]$
$[4, 10, -16, -20, 2, 22, -18, -8, -12, -6, -14]$
$[4, 10, 12, -14, 2, 22, -18, -20, -6, -8, -16]$
$[4, 10, 14, -20, 2, -16, -22, 8, -12, -6, -18]$
$[4, 10, 14, -20, 2, -16, -8, 22, -12, -6, -18]$
$[4, 10, 16, -20, 2, 22, 18, -8, 12, -6, -14]$
$[4, 12, -18, 14, 20, 2, 22, 8, 10, -6, 16]$
$[4, 14, 10, -20, 22, 18, 2, -8, 6, 12, -16]$

DT codes for the minimal diagrams, up to flype, of $K13n3370$:

[4, 12, −22, −20, −18, −16, 2, −24, −10, −8, −26, −14, −6]
[6, −10, 12, 14, −18, 20, 26, −24, −22, −4, 2, −16, −8]
[6, −10, 12, 16, −18, 20, 26, 24, −22, −4, 2, −8, 14]
[6, −10, 12, 22, −18, 20, 26, 24, −8, −4, 2, 16, 14]
[6, −10, 12, 24, −18, 20, 26, −8, −22, −4, 2, −16, 14]
[6, −10, 12, 26, −18, 20, −8, −24, −22, −4, 2, −16, −14]
[6, 10, 18, −16, 4, 24, 22, 20, −26, 2, 14, 12, −8]
[6, −10, 18, 26, 20, −2, 24, 22, 8, 4, 16, 14, 12]
[6, −10, 20, 14, −18, −4, 26, −24, −22, −12, 2, −16, −8]
[6, −10, 20, 16, −18, −4, 26, 24, −22, −12, 2, −8, 14]
[6, −10, 20, 22, −18, −4, 26, 24, −8, −12, 2, 16, 14]
[6, −10, 20, 24, −18, −4, 26, −8, −22, −12, 2, −16, 14]
[6, −10, 20, 26, −18, −4, −8, −24, −22, −12, 2, −16, −14]
[6, −10, 22, −14, −2, −20, −8, 26, 24, −12, 4, 18, 16]
[6, −12, 22, −14, −2, 20, −8, 26, 24, 4, 10, 18, 16]
[6, 14, 20, 18, −24, −16, 4, −22, −26, 2, −12, −10, −8]
[6, 14, 20, 26, 18, −16, 4, −22, −24, 2, −12, −10, 8]
[6, 14, 20, 26, −24, −16, 4, −22, −8, 2, −12, −10, 18]
[6, −14, 22, 20, 18, 26, −4, −24, 10, 8, 2, −12, −16]
[6, −14, −22, 26, 20, 18, −4, 24, 12, 10, 8, −2, 16]
[6, 20, 12, 14, −18, 4, 26, −24, −22, −10, 2, −16, −8]
[6, 20, 12, 16, −18, 4, 26, 24, −22, −10, 2, −8, 14]
[6, 20, 12, 22, −18, 4, 26, 24, −8, −10, 2, 16, 14]
[6, 20, 12, 24, −18, 4, 26, −8, −22, −10, 2, −16, 14]

DT codes for the minimal diagrams, up to flype, of $K12n288$:

[4, 10, 12, −16, 2, 8, −18, −22, −6, −24, −14, −20] [4, 10, 12, 16, 2, 8, −18, −22, 6, −24, −14, −20]
[4, 10, 12, −16, 2, 8, 20, −6, 24, 22, 14, 18] [4, 10, 14, −12, 2, 20, −18, 22, −6, 8, 24, 16]
[4, 10, 14, −12, 2, 20, −18, 22, 6, −8, 24, 16] [4, 10, −14, 16, 2, 8, 20, −24, 22, 12, 18, −6]
[4, 10, 14, −16, 2, 8, 20, −24, 22, 12, 18, −6] [4, 10, −14, 18, 2, 8, 20, −24, −22, 12, −6, −16]
[4, 10, 14, −18, 2, 20, 6, 22, 12, −8, 24, 16] [4, 10, 14, −20, 2, 8, −18, 22, −6, −12, 24, 16]
[4, 10, 14, 20, 2, 8, −18, 22, 6, −12, 24, 16] [4, 10, 14, −20, 2, 18, 24, −22, 8, 12, −16, −6]
[4, 10, 14, −22, 2, −18, 8, −24, −20, −12, −16, −6] [4, 10, −16, 12, 2, 8, −20, −6, −24, −22, −14, −18]
[4, 10, 16, 12, 2, 8, −20, 6, −24, −22, −14, −18] [4, 10, −16, 14, 2, 8, −20, 24, −22, −12, −18, −6]
[4, 10, 16, −14, 2, 8, −20, 24, −22, −12, −18, −6] [4, 10, 16, −20, 2, 18, 24, 22, 8, 12, −6, 14]
[4, 10, −18, −12, 2, 16, −20, −8, 24, −22, −14, −6] [4, 10, 18, −12, 2, 16, −20, 8, 24, −22, −14, −6]
[4, 10, 18, −14, 2, 8, −20, 24, 22, −12, −6, 16] [4, 10, 18, 16, 2, 8, −20, −22, 24, −14, −12, 6]
[4, 10, 18, −20, 2, 22, 8, −6, 24, −14, 12, 16] [4, 10, 20, −16, 2, 18, 8, −22, 12, 24, −14, −6]

DT codes for the minimal diagrams, up to flype, of $K12n491$:

[6, −12, 20, 18, 24, 16, −4, 22, 8, 2, 14, 10] [6, −16, 18, 22, 2, −4, 24, 20, −10, 12, 8, 14]
[6, 16, −18, 22, 2, −4, 24, 20, −10, 12, 8, 14] [4, 10, −16, −20, 2, −18, −22, −8, −24, −14, −6, −12]
[6, −10, 12, 22, 16, −18, 24, 20, −2, 4, 8, 14] [6, −10, 12, 22, 16, −18, 24, 20, 2, −4, 8, 14]
[6, 10, −18, 22, 2, 16, 24, 20, −4, 12, 8, 14] [6, −10, −18, 22, 16, −4, 24, 20, −2, 12, 8, 14]
[6, −10, 18, 22, 16, −4, 24, 20, 2, 12, 8, 14]

DT codes for the minimal diagrams, up to flype, of $K12n501$:

[6, −10, 22, 24, −16, 18, 20, −2, 4, 14, 12, 8] [6, −10, 22, 24, −16, 18, 20, 2, −4, 14, 12, 8]
[6, −10, 22, 24, 16, −18, −20, −4, 2, −14, −12, 8] [6, −10, 22, 24, 16, −18, −20, 4, −2, −14, −12, 8]

$[6, -12, 20, 24, 18, 16, -4, 22, 10, 2, 14, 8]$	$[4, 10, -16, -22, 2, -18, -20, -8, -24, -14, -12, -6]$
$[6, 10, 12, -18, 4, 2, -20, -22, -8, -24, -16, -14]$	$[6, 10, 12, 18, 4, 2, -20, -22, 8, -24, -16, -14]$
$[6, 10, 12, -18, 4, 2, 20, 22, 24, -8, 16, 14]$	$[6, -10, 12, 24, 16, -18, 22, 20, -2, 4, 14, 8]$
$[6, -10, 12, 24, 16, -18, 22, 20, 2, -4, 14, 8]$	$[6, -10, -14, 24, 16, -20, -18, 22, -2, -12, -4, 8]$
$[6, -10, 14, 24, 16, -20, -18, 22, 2, -12, -4, 8]$	$[6, 10, -16, 24, 2, -18, -20, -22, -4, -14, -12, 8]$
$[6, 10, 16, 24, 2, -18, -20, -22, 4, -14, -12, 8]$	$[6, -10, 16, 24, -18, -4, -20, -22, 2, -14, -12, 8]$
$[6, -10, -18, 24, 16, -4, 20, 22, -2, 14, 12, 8]$	$[6, -10, 18, 24, 16, -4, 20, 22, 2, 14, 12, 8]$

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