AUTOSTACKABILITY OF THOMPSON'S GROUP F

NATHAN CORWIN, GILI GOLAN, SUSAN HERMILLER, ASHLEY JOHNSON,
AND ZORAN ŠUNIĆ

ABSTRACT. The word problem for Thompson's group F has a solution, but it remains unknown whether F is automatic or has a finite or regular convergent (terminating and confluent) rewriting system. We show that the group F admits a natural extension of these two properties, namely autostackability, and we give an explicit bounded regular convergent prefix-rewriting system for F.

1. INTRODUCTION

The group of orientation-preserving piecewise linear automorphisms of the unit interval for which all linear slopes are powers of 2, and all breakpoints lie in the 2-adic numbers, is known as *Thompson's group F*. The group F has a well-known finite presentation

$$F = \langle x, y \mid [y, xyx^{-2}], [y, x^2yx^{-3}] \rangle$$

(and the generators x and y are the elements x_0 and x_1, respectively, of a standard infinite generating set). The group F has been the focus of considerable research in recent years, because of its connections to many other fields, because of long-standing open problems such as the amenability of F, and because of its many surprising properties (for example, F is an infinite dimensional torsion-free group with homological type FP_∞ [4]).

Many algorithmic problems have been shown to have solutions for F. The word problem is solvable for F; moreover, Guba [12] shows that the Dehn function for F is quadratic and Elder and Taback [8] show that F is 1-counter graph automatic. Guba and Sapir [13] prove that the conjugacy problem is solvable. Kassabov and Matucci [15] show that the simultaneous conjugacy problem is solvable, and Burillo, Matucci, and Ventura [5] show that the twisted conjugacy problem also has a solution. Golan and Sapir [10] give a solution to the subgroup membership problem for a large family of subgroups of F. In [1], Bleak, Brough and Hermiller show that there is an algorithm that, upon input of a finite set S of elements of F (that is, words over the generating set $A := \{x^{\pm 1}, y^{\pm 1}\}$), can determine whether or not the subgroup $\langle S \rangle$ generated by S is solvable and, if it is, also determines its derived length. In [9], Golan shows that there is an algorithm that, upon input of the finite subset $S \subset F$, can determine whether S generates the entire group F.

On the other hand, many algorithmic questions also remain open for Thompson's group F, including the questions of whether F is an automatic group (see [7] for definitions and background on automatic groups), and whether F admits a finite or regular convergent rewriting system (defined in Section 2.1); see the problem list [18]. Both of these are questions about the complexity of the word problem for F; in particular, about whether the word problem can be solved using a computer with a finite amount of memory, known as a finite state automaton.

Both (prefix-closed) automatic structures and finite convergent rewriting systems are special cases of convergent prefix-rewriting systems. A *convergent prefix-rewriting system*, or *CP-RS*, for a group G consists of a finite set A and a set $R \subset A^* \times A^*$ of ordered pairs of words over A such that G is presented, as a monoid, by

$$G = \mathrm{Mon}\langle A \mid \{u = v \mid (u, v) \in R\}\rangle,$$

and the rewritings $uz \to vz$ for all $(u, v) \in R$ and $z \in A^*$ satisfy

- (termination) there is no infinite sequence of rewritings $x_1 \to x_2 \to \cdots$, and
- (confluence) the set $\mathrm{Irr}(R)$ of words that cannot be rewritten is a set of (unique) normal forms for G.

A CP-RS is *regular* if the subset $R \subset A^* \times A^*$ is (synchronously) regular; that is, if the set of padded words obtained from R can be recognized by a finite state automaton (see Section 2.1 for more information on regular sets, finite state automata, and padded words). Otto [17] has shown that a group G has a prefix-closed automatic structure if and only if G admits a regular CP-RS R satisfying the further condition that whenever $(u, v) \in R$ then $v \in \mathrm{Irr}(R)$ and $u \in \mathrm{Irr}(R \setminus \{(u, v)\})$. The group G has a finite convergent rewriting system if and only if there is a finite set $R' \subset A^* \times A^*$ such that $\{(wu', wv') \mid w \in A^*,\ (u', v') \in R'\}$ is a CP-RS for G.

In [3], Brittenham, Hermiller, and Holt considered another solution to the word problem for groups by finite state automata that is a natural extension of both automaticity and finite rewriting systems, namely autostackability. A CP-RS R is *bounded* if there is a constant $K > 0$ such that whenever $(u, v) \in R$, then $(u, v) = (wu', wv')$ for some $w, u', v' \in A^*$ with $l(u') + l(v') \leq K$. A group G is *autostackable* if G has a bounded regular convergent prefix-rewriting system. Every group that has a prefix-closed (asynchronous or synchronous) automatic structure, or has a finite convergent rewriting system, is autostackable [3].

In analogy with the characterization of automatic groups by a regular set of normal forms with a K-fellow traveler property, autostackability of a group G with a finite inverse-closed generating set A also has a topological characterization, in terms of a discrete dynamical system on the Cayley graph $\Gamma := \Gamma_A(G)$ of G over A, as follows. A *flow function* for G with *bound* $K \geq 0$, with respect to a spanning tree T in Γ, is a function Φ mapping the set $\vec{E}$ of directed edges of Γ to the set $\vec{P}$ of directed paths in Γ, such that

- (fixing the tree) for each $e \in \vec{E}$ the path $\Phi(e)$ has the same initial and terminal vertices as e and length at most K, and Φ acts as the identity on edges lying in T, and

- (termination) there is no infinite sequence $e_1, e_2, e_3, \ldots$ of edges with each e_i not in T and each e_{i+1} in the path $\Phi(e_i)$.

Extending Φ to $\widehat{\Phi} : \vec{P} \to \vec{P}$ by $\widehat{\Phi}(e_1 \cdots e_n) := \Phi(e_1) \cdots \Phi(e_n)$, then for all $p \in \vec{P}$, there is a $n_p \in \mathbb{N}$ such that $\widehat{\Phi}^{n_p}(p)$ is a path in the tree T; that is, when $\widehat{\Phi}$ is iterated, paths in Γ "flow" toward the tree. Let $\mathcal{N}_T$ denote the set of words labeling non-backtracking paths in T starting at the vertex labeled by the identity of G, and let $\mathsf{label} : \vec{P} \to A^*$ be the function that returns the label of any directed path in Γ. The flow function Φ is *regular* if the graph of Φ, written as a subset of $A^* \times A^* \times A^*$ by

$$\mathsf{graph}\,(\Phi) := \{(\gamma, a, \mathsf{label}(\Phi(e_{\gamma,a}))) \mid \gamma \in \mathcal{N}_T, a \in A, \text{ and } e_{\gamma,a} \in \vec{E} \text{ has}$$
$$\text{initial vertex } \gamma \text{ and label } a\},$$

is regular. A group G is autostackable if and only if G admits a regular bounded flow function [3].

Several convergent prefix-rewriting systems for Thompson's group F, over the generating set $A = \{x^{\pm 1}, y^{\pm 1}\}$, have been described in earlier papers; in each case the CP-RS is computable (can be recognized by a Turing machine), but is not regular. In [6], Cleary, Hermiller, Stein and Taback show that Thompson's group F has a bounded CP-RS R (by constructing a bounded flow function) for which the set R is computable, and the normal form set $\mathrm{Irr}(R)$ is a set of words labeling quasigeodesics in the Cayley graph that is not regular, but that is a context-free language (the next simplest language class in the Chomsky hierarchy of computable languages). Guba and Sapir [11] give a convergent rewriting system R for F for which the set R is computable but not context-free (and hence not regular), but the normal form set $\mathrm{Irr}(R)$ is the regular set

$$\mathcal{N} = A^* \setminus A^* \left(\{aa^{-1} \mid a \in A\} \cup \{y^\epsilon x x^* y, y^\epsilon x^2 x^* y^{-1} \mid \epsilon \in \{1, -1\}\}\right) A^*;$$

that is, $\mathcal{N}$ is the set of all freely reduced words over A that do not contain a subword of the form $y^\epsilon x^i y$ or $y^\epsilon x^{i+1} y^{-1}$ for any $\epsilon \in \{1, -1\}$ and $i \geq 1$. Section 2.2 contains more details on the Guba-Sapir system.

In the main result of this paper we construct a new prefix-rewriting system for F using the Guba-Sapir normal form set $\mathcal{N}$, in the following.

Theorem 1.1. *Thompson's group F is autostackable. Moreover, the following is a bounded regular convergent prefix-rewriting system R for F:*

$$
\begin{aligned}
R \;=\; & \cup_{a \in A} \left\{ uaa^{-1} \to u \mid u \in A^* \right\} \\
& \cup \left(\cup_{\epsilon \in \{1,-1\}} \cup_{i \in \{1,2\}} \left\{ uy^\epsilon x^i y \to ux^i y x^{-i-1} y^\epsilon x^{i+1} \mid uy^\epsilon x^i \in \mathcal{N} \right\} \right) \\
& \cup \left(\cup_{\epsilon \in \{1,-1\}} \cup_{i \in \{2,3\}} \left\{ uy^\epsilon x^i y^{-1} \to ux^i y^{-1} x^{-i+1} y^\epsilon x^{i-1} \mid uy^\epsilon x^i \in \mathcal{N} \right\} \right) \\
& \cup \left\{ uxy \to uy^{-1} xyx^{-2} yx^2 \mid u \in \mathcal{N} \cap (A^* y^{\pm 1} A^* x^2) \right\} \\
& \cup \left\{ ux^2 y^{-1} \to uy^{-1} x^2 y^{-1} x^{-1} yx \mid u \in \mathcal{N} \cap (A^* y^{\pm 1} A^* x^2) \right\}.
\end{aligned}
$$

In essence, this system contains only finitely many rewriting rules $u' \to v'$, but when a rule can be applied is determined by the prefix, in normal form, preceding the subword on which the rule is to be applied. More particularly, for each of the

finitely many rewritings $u' \to v'$, there is a regular language $L_{(u',v')}$ such that the rewriting $wu'z \to wv'z$ can only be applied if $w \in L_{(u',v')}$. Thus the word problem for Thompson's group F can be solved by a finite state machine.

We note that the set of normal forms for the prefix-rewriting system in Theorem 1.1 is the same as the set of normal forms of the Guba-Sapir rewriting system; however, in contrast to the Guba-Sapir rules, the prefix-rewriting system in Theorem 1.1 is regular, and at each step a word of length at most 5 is replaced by a word of length at most 10. In our proof of Theorem 1.1, we consider the topological view of autostackability, and show that Thompson's group F has a regular bounded flow function.

The paper is organized as follows. We begin in Section 2 with notation used throughout the paper, background on regular languages (in Section 2.1), and a more detailed discussion of the Guba-Sapir rewriting system and its normal form set (in Section 2.2). Section 3 contains an analysis of the relationship between the normal forms (in the set $\mathcal{N}$ above) for the two endpoints of an edge of the Cayley graph $\Gamma_A(F)$ of F. In Section 4, we define a well-founded strict partial order on the edges of Γ, which is used (in Section 6) in the proof of termination for the flow function for F. In Section 5, we give a pictorial motivation for the choice of this strict partial order. Finally, Section 6 contains the proof of Theorem 1.1.

2. Background and notation

Let $x := x_0$ and $y := x_1$ be the generators of Thompson's group F, and let $A := \{x^{\pm 1}, y^{\pm 1}\}$. Let A^* be the set of all words over A. Let 1 denote the identity element of F, and let λ denote the empty word in A^*. For two words $v, w \in A^*$, we write $v = w$ if v and w are the same word in A^*, and $v =_F w$ if v and w represent the same element of F.

Let $\Gamma = \Gamma_A(F)$ be the Cayley graph of Thompson's group F over the generating set A. Given any directed edge e of Γ, let e_- and e_+ denote the elements of F at the initial and terminal vertices of e, respectively, and let $\mathsf{label}(e)$ denote the label of the edge e. We denote e by $e_{\gamma,a}$ where $a = \mathsf{label}(e)$ and $\gamma \in A^*$ is any word satisfying $\gamma =_F e_-$.

2.1. **Synchronously regular languages and rewriting systems.** Details and proofs of the contents of this section can be found in [7, 14, 3].

Let A be a finite set. The set of all finite strings over A (including the empty word λ) is written A^*. A *language* is a subset $L \subseteq A^*$. Given languages L_1, L_2 the *concatenation* $L_1 L_2$ of L_1 and L_2 is the set of all expressions of the form $l_1 l_2$ with $l_i \in L_i$. The *Kleene star* of L, denoted L^*, is the union of L^n over all integers $n \geq 0$.

The class of *regular languages* over A is the smallest class of languages that contains all finite languages and is closed under union, intersection, concatenation, complement and Kleene star. (Note that closure under some of these operations is redundant.) Regular languages are precisely those accepted by finite state automata; that is, by computers with a bounded amount of memory.

The concept of regularity is extended to subsets of a Cartesian product $(A^*)^n = A^* \times \cdots \times A^*$ of n copies of A^* as follows. Let \$ be a symbol not contained in A.

Given any tuple $w = (a_{1,1} \cdots a_{1,m_1}, ..., a_{n,1} \cdots a_{n,m_n}) \in (A^*)^n$ (with each $a_{i,j} \in A$), rewrite w to a *padded word* $\hat{w}$ over the finite alphabet $B := (A \cup \$)^n$ by $\hat{w} := (\hat{a}_{1,1}, ..., \hat{a}_{n,1}) \cdots (\hat{a}_{1,N}, ..., \hat{a}_{n,N})$ where $N = \max\{m_i\}$ and $\hat{a}_{i,j} = a_{i,j}$ for all $1 \leq i \leq n$ and $1 \leq j \leq m_i$ and $\hat{a}_{i,j} = \$$ otherwise. A subset $L \subseteq (A^*)^n$ is called a *regular language* (or, more precisely, *synchronously regular*) if the set $\{\hat{w} \mid w \in L\}$ is a regular subset of B^*.

The following lemma, much of the proof of which can be found in [7, Chapter 1], contains closure properties of regular languages that are used below in Section 6.

Lemma 2.1. *Let A be a finite set, let z be an element of A^*, and let L, L_i be regular languages over A. Then the following languages are also regular:*

(1) *(Quotient) $L_z := \{w \in A^* : wz \in L\}$.*
(2) *(Product) $L_1 \times L_2 \times \cdots \times L_n$.*

The definition of a rewriting system is very close to that of prefix-rewriting in Section 1. A *convergent rewriting system* for a group G consists of a finite set A and a set $R \subset A^* \times A^*$ of ordered pairs of words over A such that G is presented, as a monoid, by $G = \text{Mon}\langle A \mid \{u = v \mid (u, v) \in R\}\rangle$, and the rewritings $wuz \to wvz$ for all $(u, v) \in R$ and $w, z \in A^*$ satisfy the termination and confluence properties; that is, there is no infinite sequence of rewritings, and the set of irreducible words is a set of (unique) normal forms for G. (The only alteration to the definition of CP-RS needed to obtain the definition of convergent rewriting system is in the prefix w prepended in the rewritings.) The rewriting system is *finite* if R is a finite set, and the system is *regular* if the subset $R \subset A^* \times A^*$ is (synchronously) regular.

2.2. **The Guba-Sapir rewriting system for F and its normal forms.** Guba and Sapir state in [11, Theorem 2] that Thompson's group F admits a convergent rewriting system and a regular set of normal forms in the standard generators $x = x_0$ and $y = x_1$. The convergent rewriting system given in [11] is:

$$\Sigma = \left\{ \begin{array}{lll} aa^{-1} & \to & \lambda, \\ y^\epsilon x^i y & \to & x^i y x^{-i-1} y^\epsilon x^{i+1}, \\ y^\epsilon x^{i+1} y^{-1} & \to & x^{i+1} y^{-1} x^{-i} y^\epsilon x^i \end{array} \middle| \begin{array}{l} a \in \{x^{\pm 1}, y^{\pm 1}\}, \\ \epsilon \in \{1, -1\} \text{ and } i \geq 1 \end{array} \right\}.$$

We refer to the Σ-rewriting rules $aa^{-1} \to \lambda$ as free reductions, as usual, and to

$$y^\epsilon x^i y \to x^i y x^{-i-1} y^\epsilon x^{i+1}$$

and

$$y^\epsilon x^{i+1} y^{-1} \to x^{i+1} y^{-1} x^{-i} y^\epsilon x^i,$$

with $\epsilon \in \{1, -1\}$ and $i \geq 1$, as the *y-rule* and *y^{-1}-rule of size i*, respectively.

Throughout the rest of the paper, the symbol $\mathcal{N}$ will denote the set of normal forms over the generating set A defined by the rewriting system Σ; that is, $\mathcal{N}$ is the set of words that cannot be rewritten using Σ. We note that the set $\mathcal{N}$ is regular, since $\mathcal{N}$ is the set of all words over A which do not contain the left hand side of any of the rules in Σ as a subword, and since the set of left hand sides of rules in Σ is a regular language. However, the appearance of the integer i three times on the right hand

side of the y- and y^{-1}-rules of the rewriting system Σ shows that the set of padded words obtained from Σ does not obey the Pumping Lemma for regular languages [14, Lemma 3.1], nor the Pumping Lemma for context-free languages [14, Lemma 6.1], and so the rewriting system Σ is neither (synchronously) regular nor context-free.

The following lemma is immediate from the description of Σ.

Lemma 2.2. *The set of normal forms $\mathcal{N}$ is the set of all words over A of the form*

$$x^{i_n} y^{\epsilon_n} \cdots x^{i_1} y^{\epsilon_1} x^{i_0},$$

such that $n \geq 0$, each $\epsilon_j \in \{\pm 1\}$, each $i_j \in \mathbb{Z}$, and for every $j \in \{1, \ldots, n-1\}$ the following hold.

(1) *If $i_j = 0$ then $\epsilon_j = \epsilon_{j+1}$ (i.e., the word is freely reduced).*
(2) *If $\epsilon_j = 1$ then $i_j \leq 0$.*
(3) *If $\epsilon_j = -1$ then $i_j \leq 1$.*

For any element $g \in F$, write $\mathsf{nf}(g)$ for the normal form in $\mathcal{N}$ representing G; similarly, for any word w over A, write $\mathsf{nf}(w)$ for the unique word in $\mathcal{N}$ such that $w =_F \mathsf{nf}(w)$.

Throughout this paper we also let T denote the subtree of Γ associated with $\mathcal{N}$; in other words, T is the subtree of Γ such that the set of labels of nonbacktracking paths in T, starting from the identity vertex, is the set of normal forms in $\mathcal{N}$.

Next we give a description of the edges that do not lie in the tree T. Note that a directed edge $e = e_{\gamma,a}$ with $\gamma = \mathsf{nf}(e_-)$ lies on the tree T if and only if either $\gamma a \in \mathcal{N}$ or the last letter of γ is a^{-1}. That is, the edge $e = e_{\gamma,a}$ lies on T if and only if the freely reduced word freely equivalent to γa belongs to $\mathcal{N}$. Hence Lemma 2.2 implies the following.

Lemma 2.3. *Let $e = e_{\gamma,a}$ be a directed edge of Γ where $a \in \{x^{\pm 1}, y^{\pm 1}\}$ and*

$$\gamma = x^{i_n} y^{\epsilon_n} \cdots x^{i_1} y^{\epsilon_1} x^{i_0} \in \mathcal{N}.$$

Then e does not lie on the tree T if and only if one of the following holds.

(1) $\mathsf{label}(e) = y$, $n \geq 1$, and $i_0 \geq 1$.
(2) $\mathsf{label}(e) = y^{-1}$, $n \geq 1$, and $i_0 \geq 2$.

In particular, all edges of Γ labeled by $x^{\pm 1}$ lie on the tree T.

We end this section by defining a few parameters associated to every word in normal form.

Let $\gamma = x^{i_n} y^{\epsilon_n} \cdots x^{i_1} y^{\epsilon_1} x^{i_0}$ be a word in normal form. Denote by $s(\gamma)$ the vector $s(\gamma) = (s_n, \ldots, s_0)$ of cumulative x-exponents in γ defined, for $k = 0, \ldots, n$, by

$$s_k := i_k + i_{k-1} + \cdots + i_0.$$

Note that the vector has at least one entry, even when the word γ is empty (indeed, in that case, $s(\gamma) = (0)$). We also define two cutoff points, namely, let

$$m(\gamma) := \min\{k \mid s_k \leq 0\},$$

if the set on the right hand side is nonempty; otherwise, let $m(\gamma) := n$, and similarly,

$$m'(\gamma) := \min\{k \mid s_k \leq 1\},$$

if the set on the right hand side is nonempty; otherwise, let $m'(\gamma) := n$. We extend our notation to elements g of F by declaring $s(g)$ to mean $s(\mathsf{nf}(g))$, $m(g)$ to mean $m(\mathsf{nf}(g))$, and so on.

3. Normal forms of the endpoints of edges outside of the normal form tree

We now turn to the analysis of the relation between the normal forms of the endpoints of edges that are not on the normal form tree T. It will be convenient to set up and fix a situation and notation used throughout Sections 3 and 4.

Let e be an edge not on the tree T, labeled by y, connecting g to g'; that is,

$$e : \overset{g}{\bullet} \overset{y}{\longrightarrow} \overset{g'}{\bullet}$$

satisfies $e_- = g$, $e_+ = g'$, and $g' =_F gy$. We also consider the inverse edge e^{-1}, labeled by y^{-1}, directed from g' to g. Let the normal forms of the endpoints of e be given by

$$\gamma := \mathsf{nf}(g) = x^{i_n} y^{\epsilon_n} \cdots x^{i_1} y^{\epsilon_1} x^{i_0},$$

with each $\epsilon_\ell \in \{\pm 1\}$, and

$$\gamma' := \mathsf{nf}(g') = x^{j_{n'}} y^{\varepsilon_{n'}} \cdots x^{j_1} y^{\varepsilon_1} x^{j_0},$$

with each $\varepsilon_\ell \in \{\pm 1\}$. Note that, along with the usual requirements from Lemma 2.2 that the normal forms γ and γ' must satisfy, we also have the additional requirements from Lemma 2.3, since e is not on the tree T. Thus, $n, n' \geq 1$, $i_0 \geq 1$ and $j_0 \geq 2$.

Using the notation from Section 2.2, let $m := m(\gamma)$ and $m' := m'(\gamma')$. Then

$$s(\gamma) = (s_n, \ldots, s_0), \qquad s(\gamma') = (s'_{n'}, \ldots, s'_0),$$

$$m = m(\gamma) = \min\{k \mid s_k \leq 0\}, \qquad m' = m'(\gamma') = \min\{k \mid s'_k \leq 1\}.$$

Since the rewriting system Σ is convergent, there is a sequence of Σ-rewritings from any word in A^* to its normal form; indeed, there may be many different such derivations. We define the *standard Σ-rewriting*, or *standard Σ-derivation*, of a word in A^* to be the sequence of Σ-rewritings to the normal form where at each rewriting step a Σ-rule is applied to the shortest possible rewritable prefix. In the following subsections we provide details of the standard Σ-rewritings of γy and $\gamma' y^{-1}$ to their normal forms γ' and γ, respectively. We also extract some parameters associated to those standard Σ-rewritings directly from γ and γ', and more specifically, from $s(\gamma)$, $s(\gamma')$, m, and m'.

3.1. **Rewriting γy to γ'.** In this section we describe how to obtain the normal form γ' for the terminal vertex $e_+ =_F \gamma y$ of $e = e_{\gamma,y}$, starting from the word γy.

Lemma 3.1. *We have $m \geq 1$, and the following assertions hold.*

(1) *If either $m = n$ or $s_m \neq 0$ or $y^{\epsilon_{m+1}} = y$, then*

$$\gamma' = \mathsf{nf}(\gamma y) = x^{i_n} y^{\epsilon_n} \cdots x^{i_{m+1}} y^{\epsilon_{m+1}} x^{s_m} y x^{-s_{m-1}-1} y^{\epsilon_m} x^{i_{m-1}} y^{\epsilon_{m-1}} \cdots x^{i_1} y^{\epsilon_1} x^{i_0+1}.$$

(2) *Otherwise, if $m < n$, $s_m = 0$, and $y^{\epsilon_{m+1}} = y^{-1}$, then*

$$\gamma' = \mathsf{nf}(\gamma y) = x^{i_n} y^{\epsilon_n} \cdots x^{i_{m+2}} y^{\epsilon_{m+2}} x^{i_{m+1}+i_m-1} y^{\epsilon_m} x^{i_{m-1}} y^{\epsilon_{m-1}} \cdots x^{i_1} y^{\epsilon_1} x^{i_0+1}.$$

In each case, there is a sequence of Σ-rewriting rules leading from γy to its normal form γ' such that no y^{-1}-rules are ever used, exactly m y-rules are used and their sizes are, in the order in which they are used,

$$s_0, \ s_1, \ s_2, \ \ldots, \ s_{m-1}.$$

(*In item (1), the prefix $x^{i_n} y^{\epsilon_n} \cdots x^{i_{m+1}} y^{\epsilon_{m+1}}$ is omitted when $m = n$, and in item (2), the prefix $x^{i_n} y^{\epsilon_n} \cdots x^{i_{m+2}} y^{\epsilon_{m+2}}$ is omitted when $m = n - 1$.*)

Proof. Since $s_0 = i_0 \geq 1$, we have $m \geq 1$.

Our standard Σ-derivation proceeds from γy in m phases. During each of the m phases exactly one y-rule is used followed by a complete free reduction. In detail, phases 1 through $m - 1$ proceed as follows. In phase 1 the y-rule of size s_0 is applied to $y^{\epsilon_1} x^{s_0} y$, followed by a complete free reduction of the obtained word. Then, in phase 2 the y-rule of size s_1 is applied to $y^{\epsilon_2} x^{s_1} y$, followed by a complete free reduction, and so on, until in phase $m - 1$ the y-rule of size s_{m-2} is applied to $y^{\epsilon_{m-1}} x^{s_{m-2}} y$, followed by a complete free reduction. In phase m, the y-rule of size s_{m-1} is applied to $y^{\epsilon_m} x^{s_{m-1}} y$, followed by free reduction (not necessarily complete), yielding the word

$$(1) \qquad\qquad x^{s_m} y x^{-s_{m-1}-1} y^{\epsilon_m} x^{i_{m-1}} y^{\epsilon_{m-1}} \cdots x^{i_1} y^{\epsilon_1} x^{i_0+1},$$

if $n = m$ or, otherwise, if $n > m$,

$$(2) \qquad x^{i_n} y^{\epsilon_n} \cdots \underline{x^{i_{m+1}} y^{\epsilon_{m+1}} x^{s_m} y x^{-s_{m-1}-1}} y^{\epsilon_m} x^{i_{m-1}} y^{\epsilon_{m-1}} \cdots x^{i_1} y^{\epsilon_1} x^{i_0+1}.$$

At this point, to finish phase m of the standard Σ-rewriting to γ' we just need to complete the free reduction.

Let us indicate why the rewriting described up to this point (before this last free reduction) is actually possible; that is, why we eventually arrive, during phase m, at a word of the form (1) or (2) in the appropriate cases.

The original word

$$\gamma y = x^{i_n} y^{\epsilon_n} \cdots x^{i_1} y^{\epsilon_1} x^{i_0} y = x^{i_n} y^{\epsilon_n} \cdots x^{i_1} \underline{y^{\epsilon_1} x^{s_0} y}$$

is freely reduced and since its prefix γ is a normal form, the only available Σ-rewriting is the application of the y-rule of size s_0 to the underlined portion of the word, yielding a word that freely reduces to

$$x^{s_1} y x^{-s_0-1} y^{\epsilon_1} x^{i_0+1},$$

if $n = m = 1$, and to

$$x^{i_n} y^{\epsilon_n} \cdots x^{i_2} \underline{y^{\epsilon_2} x^{s_1} y} x^{-s_0-1} y^{\epsilon_1} x^{i_0+1},$$

in all other cases. If $m = 1$ we are done (we arrived at a word of the form (1) or (2) during phase 1).

If $n \geq m \geq 2$, then we are in the second case above. The prefix $y^{\epsilon_n} \cdots y^{\epsilon_2}$ is nonempty and y^{ϵ_2} actually appears in it. Note also that $s_0 \geq 1$, which implies that $-s_0 - 1 \leq -2$ and, since $m \geq 2$, $s_1 \geq 1$. Thus the word is freely reduced and the only available Σ-rewriting is the application of the y-rule of size s_1 to the underlined portion of the word, yielding a word that freely reduces to

$$x^{s_2} y x^{-s_1-1} y^{\epsilon_2} x^{i_1} y^{\epsilon_1} x^{i_0+1},$$

if $n = m = 2$, and to

$$x^{i_n} y^{\epsilon_n} \cdots x^{i_3} \underline{y^{\epsilon_3} x^{s_2} y} x^{-s_1-1} y^{\epsilon_2} x^{i_1} y^{\epsilon_1} x^{i_0+1},$$

in all other cases. If $m = 2$ we are done (we arrived at a word of the form (1) or (2) during phase 2).

If $n \geq m \geq 3$, then we are in the second case above. The portion of the word $y^{\epsilon_n} \cdots y^{\epsilon_3}$ is nonempty and y^{ϵ_3} actually appears in it. Note also that $s_1 \geq 1$, which implies that $-s_1 - 1 \leq -2$ and, since $m \geq 3$, $s_2 \geq 1$. Thus the word is freely reduced and the only available Σ-rewriting is the application of the y-rule of size s_2 to the underlined portion of the word... Et cetera; we continue this process through the m phases.

Let us consider now the words (1) or (2) obtained during phase m.

Note that, since $s_{m-1} \geq 1$, we have $-s_{m-1} - 1 \leq -2$. Therefore, if $n = m$, the word (1) is freely reduced and, since no $y^{\pm}$ rules are applicable, it is the normal form of γy. In the other case, $n > m$, we obtain the word (2). If $s_m \neq 0$ or $y^{\epsilon_{m+1}} = y$ this word is also freely reduced and it is the normal form of γy. Finally, if $n > m$, $s_m = 0$, and $y^{\epsilon_{m+1}} = y^{-1}$, then there are further free reductions in the underlined portion of the word (2), yielding the word

$$x^{i_n} y^{\epsilon_n} \cdots x^{i_{m+2}} y^{\epsilon_{m+2}} x^{i_{m+1}-s_{m-1}-1} y^{\epsilon_m} x^{i_{m-1}} y^{\epsilon_{m-1}} \cdots x^{i_1} y^{\epsilon_1} x^{i_0+1}.$$

We claim that, in this last case, either $n = m + 1$ or $i_{m+1} - s_{m-1} - 1 \leq -1$, which implies that the obtained word is freely reduced and, since no $y^{\pm}$-rules can be applied, it is the normal form of γy. Indeed, if $n > m + 1$ then $-s_{m-1} \leq -1$ (since $s_{m-1} \geq 1$) and $i_{m+1} \leq 1$, which yields $i_{m+1} - s_{m-1} - 1 \leq -1$. Since $s_m = 0$, then $-s_{m-1} = i_m$, completing the proof of part (2) of the lemma. $\qquad\square$

Remark 3.2. It will be useful to record some of the conclusions of Lemma 3.1 in a slightly different form. Namely, if either $m = n$ or $s_m \neq 0$ or $y^{\epsilon_{m+1}} = y$, then there are no free cancellations of y letters during the standard Σ-rewriting, $n' = n + 1$, and the x-exponents and the cumulative x-exponents of γ' are given in the left half of Table 1. We call this case the *no y-cancellation case*. Otherwise, if $m < n$, $s_m = 0$, and $y^{\epsilon_{m+1}} = y$, then there is a single free cancellation of $y^{\pm}$-letters during the standard Σ-rewriting, $n' = n - 1$ (note that, in this case $n \geq m + 1 \geq 2$), and the x-exponents and the cumulative x-exponents of γ' are given in the right half of Table 1. We call this latter case the *y-cancellation case*.

k	j_k	s'_k
0	$i_0 + 1$	$s_0 + 1$
1	i_1	$s_1 + 1$
$\vdots$	$\vdots$	$\vdots$
$m-1$	i_{m-1}	$s_{m-1} + 1$
m	$-s_{m-1} - 1$	0
$m+1$	s_m	s_m
$m+2$	i_{m+1}	s_{m+1}
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
$n+1$	i_n	s_n

k	j_k	s'_k
0	$i_0 + 1$	$s_0 + 1$
1	i_1	$s_1 + 1$
$\vdots$	$\vdots$	$\vdots$
$m-1$	i_{m-1}	$s_{m-1} + 1$
m	$i_{m+1} + i_m - 1$	s_{m+1}
$m+1$	i_{m+2}	s_{m+2}
$\vdots$	$\vdots$	$\vdots$
$n-1$	i_n	s_n

TABLE 1. x-exponents and cumulative x-exponents of γ': the no y-cancellation case (left) and the y-cancellation case (right)

3.2. **Rewriting $\gamma'y^{-1}$ to γ.** The previous lemma describes, in much detail, how to "multiply by y". The next lemma, regarding "multiplication by y^{-1}", has a similar statement and an analogous proof, so we skip some but not all details, since there are a few subtle points that are sufficiently different.

Lemma 3.3. *We have $m' \geq 1$, and the following assertions hold.*

(1) *If either $m' = n'$ or $s'_{m'} \neq 0$ or $y^{\varepsilon_{m'+1}} = y^{-1}$, then*

$$\gamma = \mathsf{nf}(\gamma'y^{-1}) = x^{j_{n'}}y^{\varepsilon_{n'}} \cdots x^{j_{m'+1}}y^{\varepsilon_{m'+1}}x^{s'_{m'}}y^{-1}x^{-s'_{m'-1}+1}y^{\varepsilon_{m'}}x^{j_{m'-1}}y^{\varepsilon_{m'-1}} \cdots x^{j_1}y^{\varepsilon_1}x^{j_0-1}.$$

(2) *Otherwise, if $m' < n'$, $s'_{m'} = 0$, and $y^{\varepsilon_{m'+1}} = y$, then*

$$\gamma = \mathsf{nf}(\gamma'y^{-1}) = x^{j_{n'}}y^{\varepsilon_{n'}} \cdots x^{j_{m'+2}}y^{\varepsilon_{m'+2}}x^{j_{m'+1}+j_{m'+1}}y^{\varepsilon_{m'}}x^{j_{m'-1}}y^{\varepsilon_{m'-1}} \cdots x^{j_1}y^{\varepsilon_1}x^{j_0-1}.$$

In each case, there is a sequence of Σ-rewriting rules leading from $\gamma'y^{-1}$ to its normal form γ such that no y-rules are ever used, exactly m' y^{-1}-rules are used and their sizes are, in the order in which they are used,

$$s'_0 - 1, \; s'_1 - 1, \; s'_2 - 1, \; \ldots \; , \; s'_{m'-1} - 1.$$

(In item (1), the prefix $x^{j_{n'}}y^{\varepsilon_{n'}} \cdots x^{j_{m'+1}}y^{\varepsilon_{m'+1}}$ is omitted when $m' = n'$, and in item (2), the prefix $x^{j_{n'}}y^{\varepsilon_{n'}} \cdots x^{j_{m'+2}}y^{\varepsilon_{m'+2}}$ is omitted when $m' = n' - 1$.)

Proof. Since $s'_0 = j_0 \geq 2$, we have $m' \geq 1$.

Our standard Σ-derivation proceeds from $\gamma'y^{-1}$ in m' phases. During each of the m' phases exactly one y^{-1}-rule is used followed by a complete free reduction. In detail, phases 1 through $m' - 1$ proceed as follows. In phase 1 the y^{-1}-rule of size $s'_0 - 1$ is applied to $y^{\varepsilon_1}x^{s'_0}y^{-1}$, followed by a complete free reduction of the obtained word. Then, in phase 2 the y^{-1}-rule of size $s'_1 - 1$ is applied to $y^{\varepsilon_2}x^{s'_1}y^{-1}$, followed by a complete free reduction, and so on, until in phase $m' - 1$ the y^{-1}-rule of size $s'_{m'-2} - 1$ is applied to $y^{\varepsilon_{m'-1}}x^{s'_{m-2}}y^{-1}$, followed by a complete free reduction. In phase m', the

y^{-1}-rule of size $s'_{m'-1} - 1$ is applied to $y^{\varepsilon_{m'}} x^{s'_{m'-1}} y^{-1}$, followed by free reduction (not necessarily complete), yielding the word

$$(3) \qquad x^{s'_{m'}} y^{-1} x^{-s'_{m'-1}+1} y^{\varepsilon_{m'}} x^{j_{m'-1}} y^{\varepsilon_{m'-1}} \cdots x^{j_1} y^{\varepsilon_1} x^{j_0-1},$$

if $n' = m'$ or, otherwise, the word

$$(4) \qquad x^{j_{n'}} y^{\varepsilon_{n'}} \cdots \underline{x^{j_{m'+1}} y^{\varepsilon_{m'+1}} x^{s'_{m'}} y^{-1} x^{-s'_{m'-1}+1}} y^{\varepsilon_{m'}} x^{j_{m'-1}} y^{\varepsilon_{m'-1}} \cdots x^{j_1} y^{\varepsilon_1} x^{j_0-1}.$$

Note that, since $s'_{m'-1} \geq 2$, we have $-s'_{m'-1} + 1 \leq -1$. Therefore, if $n' = m'$, the word (3) is freely reduced and, since no $y^{\pm}$ rules are applicable, it is the normal form of $\gamma' y^{-1}$. In the other case, $n' > m'$, we obtain the word (4). If $s'_{m'} \neq 0$ or $y^{\varepsilon_{m'+1}} = y^{-1}$ this word is also freely reduced and it is the normal form of $\gamma' y^{-1}$. Finally, if $n' > m'$, $s'_{m'} = 0$, and $y^{\varepsilon_{m'+1}} = y$, then there are further free reductions in the underlined portion of the word (4), yielding the word

$$x^{j_{n'}} y^{\varepsilon_{n'}} \cdots x^{j_{m'+2}} y^{\varepsilon_{m'+2}} x^{j_{m'+1}-s'_{m'-1}+1} y^{\varepsilon_{m'}} x^{j_{m'-1}} y^{\varepsilon_{m'-1}} \cdots x^{j_1} y^{\varepsilon_1} x^{j_0-1}.$$

We claim that, in this last case, either $n' = m' + 1$ or $j_{m'+1} - s'_{m'-1} + 1 \leq -1$, which implies that the obtained word is freely reduced and, since no $y^{\pm}$-rules can be applied, it is the normal form of $\gamma' y^{-1}$. Indeed, if $n' > m' + 1$ then $s'_{m'-1} \leq -2$ (since $s'_{m'-1} \geq 2$) and $j_{m'+1} \leq 0$ (since $\varepsilon_{m'+1} = 1$), which yields $j_{m'+1} - s'_{m'-1} + 1 \leq -1$. Since $s'_{m'} = 0$, then $-s'_{m'-1} = j_{m'}$, completing the proof of part (2) of the lemma. $\square$

Remark 3.4. It will also be useful to record some of the conclusions of Lemma 3.3 in a slightly different form. Namely, if either $m' = n'$ or $s'_{m'} \neq 0$ or $y^{\epsilon_{m'+1}} = y$, then there are no free cancellations of y letters during the standard Σ-rewriting, $n = n' + 1$, and the x-exponents and the cumulative x-exponents of γ are given in the left half of Table 2. We call this case the *no y^{-1}-cancellation case*. Otherwise, if $m' < n'$,

k	i_k	s_k
0	$j_0 - 1$	$s'_0 - 1$
1	j_1	$s'_1 - 1$
$\vdots$	$\vdots$	$\vdots$
$m' - 1$	$j_{m'-1}$	$s'_{m'-1} - 1$
m'	$-s'_{m'-1} + 1$	0
$m' + 1$	$s'_{m'}$	$s'_{m'}$
$m' + 2$	$j_{m'+1}$	$s'_{m'+1}$
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
$n' + 1$	$j_{n'}$	$s'_{n'}$

k	i_k	s_k
0	$j_0 - 1$	$s'_0 - 1$
1	j_1	$s'_1 - 1$
$\vdots$	$\vdots$	$\vdots$
$m' - 1$	$j_{m'-1}$	$s'_{m'-1} - 1$
m'	$j_{m'+1} + j_{m'} + 1$	$s'_{m'+1}$
$m' + 1$	$j_{m'+2}$	$s'_{m'+2}$
$\vdots$	$\vdots$	$\vdots$
$n' - 1$	$j_{n'}$	$s'_{n'}$

TABLE 2. x-exponents and cumulative x-exponents of γ': the no y^{-1}-cancellation case (left) and the y^{-1}-cancellation case (right)

$s'_{m'} = 0$, and $y^{\epsilon_{m'+1}} = y$, then there is a single free cancellation of $y^{\pm}$-letters during

the standard Σ-rewriting, $n = n' - 1$, and the x-exponents and the cumulative x-exponents of γ are given in the right half of Table 2. We call this latter case the y^{-1}-*cancellation case*.

In the Lemmas 3.1 and 3.3, we produced standard Σ-rewritings of γy and $\gamma' y^{-1}$ when the edges corresponding to y and y^{-1}, respectively, are not on the normal form tree. We do not need the following observation, but a careful reading of the proofs reveals that any Σ-rewriting of γy uses m y-rules of the same sizes and in the same order as in the standard rewriting, and uses no y^{-1}-rules. Similarly, any Σ-rewriting of $\gamma' y^{-1}$ uses m' y^{-1}-rules of the same sizes and in the same order as in the standard rewriting, and uses no y-rules. In other words, the only freedom of choice we have during the Σ-rewriting of γy and $\gamma' y^{-1}$ is in the order of performing the free reductions (including the possibility of postponing some free reductions and applying some $y^{\pm}$-rules before the word is freely reduced).

4. Ordering the directed edges in the Cayley graph

In this section we define a well-founded strict partial order on the edges of the Cayley graph Γ of F, which will be used in Section 6 in the proof of the termination of the flow function.

Throughout this section we use the same notation defined in Section 3. In particular, e is a directed edge labeled by y, the initial vertex g of e has normal form γ and the terminal vertex has normal form γ', the vectors of cumulative x-exponents are $s(\gamma) = (s_n, \ldots, s_0)$ and $s(\gamma') = (s'_{n'}, \ldots, s'_0)$, and the cutoff points are $m := m(\gamma)$ and $m' := m'(\gamma')$.

4.1. **Size sequence of an edge.** In this subsection we define the size sequence of a directed edge e of the Cayley graph Γ of Thompson's group F that does not lie in the tree T determined by the set $\mathcal{N}$ of normal forms.

We begin by showing that, not surprisingly, the Σ-rewritings corresponding to pairs of inverse edges are closely related.

Lemma 4.1. *We have $m = m'$ and the sequence of y-rule sizes used in the rewriting of γy is the same (including the order) as the sequence of y^{-1}-rule sizes in the rewriting of $\gamma' y^{-1}$.*

Proof. From Table 1 we have $s'_k = s_k + 1 \geq 1 + 1 = 2$, for $k = 0, \ldots, m - 1$, and so $m' \geq m$. Similarly, from Table 2 we have $s_k = s'_k - 1 \geq 2 - 1 = 1$, for $k = 0, \ldots, m' - 1$, and so $m \geq m'$. Hence $m = m'$. We see from Lemma 3.3 that the sizes of the y^{-1}-rules used in the standard Σ-rewriting of $\gamma' y^{-1}$ to γ are, for $k = 0, \ldots, m - 1$ (thus, in order in which they are used), $s'_k - 1 = s_k$, agreeing with the ordered list of sizes in the rewritings from γy to γ' in Lemma 3.1. $\square$

We define the *size sequence* of the edge e labeled by y to be the sequence

$$\sigma(e) = (s_0, \ldots, s_{m-1})$$

of y-rule sizes used to "cross" the edge e, that is, to rewrite γy to γ'. Note that both the length $m(\gamma)$ of this sequence, denoted from now on also by $m(e)$, and its

members can be easily "read" from the x-exponents of the normal form of the vertex $g = e_-$. Similarly, we define the size sequence of the edge e^{-1} labeled by y^{-1} to be the sequence

$$\sigma(e^{-1}) = (s'_0 - 1, \ldots, s'_{m'-1} - 1)$$

of y^{-1}-rule sizes used to "cross" the edge e^{-1} that is, to rewrite $\gamma' y^{-1}$ to γ. Note that both the length $m'(\gamma')$ of this sequence, denoted from now on also by $m(e^{-1})$, and its members can be easily "read" from the x-exponents of the normal form of the vertex $g' = (e^{-1})_-$. Moreover, by Lemma 4.1, these two sequences coincide, that is, they have the same length $m(e) = m(e^{-1})$ and exactly the same members $\sigma(e) = \sigma(e^{-1})$. Therefore, the size sequence of an edge can be easily read from the vertex of either end of the edge, by using either Table 1 or Table 2, as convenient and appropriate.

4.2. **Weight and order on edges outside of the tree of normal forms.** In this subsection we define a well-founded strict partial order on the directed edges of the Cayley graph Γ of F over the generating set $A = \{x^{\pm 1}, y^{\pm 1}\}$. A well-founded strict partial order is a strict partial order for which there is no infinite descending chain; such orders are useful in proving termination of (prefix-)rewriting systems.

The order will be defined by simply using a weight function on the edges. The weight function uses the size sequence of the edge and the following sequence.

Let C be the sequence of positive integers defined recursively by

$$C(1) = 1,$$
$$C(2) = 1,$$
$$C(i) = C(i-1) + 2C(i-2) + 2, \text{ for } i \geq 3.$$

The members of the sequence are explicitly given by

$$C(i) = \frac{2}{3} \cdot 2^i - \frac{2}{3}(-1)^i - 1,$$

but the only important features of the sequence C for our purposes are that it is nondecreasing sequence of positive integers such that $C(i) > C(i-1) + C(1)$ for all $i \geq 3$. Of course, formally speaking, this means that we could have chosen another, perhaps simpler, sequence than C to define the weights, but we have chosen C because it actually has meaningful interpretation within our problem, described further in Section 5.

Definition 4.2 (Edge weight). Let e be a directed edge in Γ that is not on the normal form tree T, with size sequence $\sigma(e) = (s_0, \ldots, s_{m-1})$ of length $m = m(e)$. The *weight* $W(e)$ of the edge e is the positive integer

$$W(e) = \sum_{k=0}^{m-1} C(s_k).$$

Definition 4.3 (Edge order). We define a strict partial order on the set of edges not on the normal form tree by

$$e \prec e' \iff W(e) < W(e').$$

The relation $\prec$ defined above is indeed a well-founded strict partial order, since the relation $<$ on the natural numbers is a well-founded strict partial order. Note also that Lemma 4.1 shows that $W(e) = W(e^{-1})$ for any edge e.

5. Interlude: Motivation for the definition of the weight function

In this section we give a pictorial, diagrammatic motivation for the weight function defined in Section 4.2. We note that the rest of the paper, including the proof of Theorem 1.1 in Section 6, does not require the material in this section; this section is provided to explain how the proof came about.

5.1. Background on van Kampen diagrams.
Let $\mathcal{P} = \langle A \mid R \rangle$ be a presentation for a group G, with the generating set A closed under inversion. Let $w \in A^*$ be an arbitrary word that represents the trivial element 1 of G. A *van Kampen diagram* Δ for w with respect to $\mathcal{P}$ is a finite, planar, contractible combinatorial 2-complex with edges directed and labeled by elements of A, satisfying the properties that the boundary of Δ is an edge path labeled by the word w starting at a basepoint vertex $*$ and reading counterclockwise, and every 2-cell in Δ has boundary labeled by an element of R.

Since $w =_G 1$, there is a factorization of w in the free group $F(A)$ as a product of conjugates of elements of the relator set R. Such a factorization gives rise to a van Kampen diagram for w, with the same number of 2-cells as factors. In general, there may be many different van Kampen diagrams for the word w.

See for example [2] or [16] for an exposition of the theory of van Kampen diagrams.

5.2. Filled box diagrams for Thompson's group F.
We consider the generating set $A = \{x^{\pm 1}, y^{\pm 1}\}$ of F, and construct van Kampen diagrams using two presentations of F with this generating set, namely

$$\mathcal{P} \; := \; \langle A \mid [y, xyx^{-2}], [y, x^2yx^{-3}]\rangle,$$
$$\mathcal{P}' \; := \; \langle A \mid \{[y, x^i yx^{-i-1}] \mid i \geq 1\}\rangle.$$

As in previous sections, let e be a directed edge, with label y, in the Cayley graph $\Gamma = \Gamma_A(F)$ that does not lie in the tree T determined by the normal form set $\mathcal{N}$. Let

$$\gamma := \mathsf{nf}(e_-) = x^{i_n}y^{\epsilon_n} \cdots x^{i_1}y^{\epsilon_1}x^{i_0}$$

and $\gamma' := \mathsf{nf}(e_+)$. The vector of cumulative x-exponents for γ is $s(\gamma) = (s_n, \ldots, s_0)$, the cutoff point is $m := m(\gamma)$, and the size sequence is $\sigma(e) = (s_0, \ldots, s_{m-1})$.

Consider the word $z = z(e) := \gamma y(\gamma')^{-1}$ in A^*, where $(\ )^{-1}$ denotes a formal inverse. The word z represents the identity element of F, and so for each of the presentations of F with generating set A, there is a van Kampen diagram for z. We describe a procedure for building a specific van Kampen diagram Δ_z for z with respect to the presentation $\mathcal{P}$ as follows.

First we use the analysis in Section 3.1 to build a van Kampen diagram Δ'_z for z over the presentation $\mathcal{P}'$. Each of the m phases of the standard Σ-rewriting from the word γy to the word γ' consists of applying a single defining relation from the presentation $\mathcal{P}'$. In particular, for $0 \leq k \leq m-1$, the relation applied is $[y, x^{s_k}yx^{-s_k-1}]$. We

draw the 2-cell for this relation as a rectangle, with top and bottom labeled $y^{\epsilon_{k+1}}$ and pointing to the right, with left and right sides labeled $x^{s_k}yx^{-s_k-1}$ and oriented from bottom to top. The diagram Δ'_z consists of a 1-dimensional edge path for a common prefix of γy and γ', together with these m rectangles, or "boxes", in a horizontal sequence with the right side of the k-th 2-cell attached to the left side of the $(k-1)$-th 2-cell along the vertical y edges, also gluing as many x and x^{-1} edges as possible along the two vertical sides. An illustration of the diagram Δ'_z in the case that $\gamma = x^2y^{-1}xy^{-1}x^{-2}yx^4$ is given in Figure 1.

FIGURE 1. Δ'_z with $\gamma = x^2y^{-1}xy^{-1}x^{-2}yx^4$

Next we build an iterative procedure for filling a rectangular 2-cell labeled by a relator of the presentation $\mathcal{P}'$ with a van Kampen diagram over the presentation $\mathcal{P}$. Any rectangular box labeled by the relator $[y, x^iyx^{-i-1}]$ with $i \in \{1, 2\}$ of $\mathcal{P}'$ is also a 2-cell labeled by a relator of $\mathcal{P}$, and so is left unchanged. For any $i \geq 3$, the rectangular box labeled by the relator $[y, x^iyx^{-i-1}]$ can be filled using two 2-cells with boundary labels $[y, xyx^{-2}]$, two copies of the van Kampen diagram over $\mathcal{P}$ for the word $[y, x^{i-2}yx^{-(i-2)-1}]$, and one copy of the van Kampen diagram over $\mathcal{P}$ for the word $[y, x^{i-1}yx^{-(i-1)-1}]$; see Figure 2 for an illustration of the resulting van Kampen diagram over $\mathcal{P}$ for the word $[y, x^iyx^{-i-1}]$.

Finally, the van Kampen diagram Δ_z for the word $z = \gamma y(\gamma')^{-1}$ is obtained from the diagram Δ'_z by replacing each 2-cell with the van Kampen diagram over the presentation $\mathcal{P}$ built by this iterative procedure. We refer to the diagram Δ_z as the *filled box diagram for the edge e*.

We next count the number of 2-cells in the filled box diagram Δ_z for e. Note that this box diagram may not be reduced; that is, there may be adjacent 2-cells in Δ_z that are labeled by the same relator with opposite orientations. Hence we may not have a van Kampen diagram of minimal possible area (number of 2-cells) for the word z; we simply have a diagram that is built in a canonical way.

For all $i \geq 1$, let $C(i)$ denote the number of 2-cells in the van Kampen diagram over $\mathcal{P}$ for the word $[y, x^iyx^{-i-1}]$ built by our iterative procedure. Then $C(1) = 1$, $C(2) = 1$, and for all $i \geq 3$ we have $C(i) = C(i-1) + 2C(i-2) + 2$. That is, the numbers $C(i)$ give the sequence of positive integers defined in Section 4.2. Now the

FIGURE 2. Van Kampen diagram for the relator $[y, x^i y x^{-i-1}]$ with $i \geq 3$

number of 2-cells in the filled box diagram Δ_z associated to the edge e is the sum of the numbers of 2-cells from presentation $\mathcal{P}$ that are used to fill the rectangles in the diagram Δ'_z; that is, the number of 2-cells in the filled box diagram Δ_z is

$$\sum_{k=0}^{m-1} C(s_k),$$

which is our definition of the weight of e in Section 4.2.

6. THOMPSON'S GROUP F IS AUTOSTACKABLE

In this final section we prove that Thompson's group F is autostackable. As in earlier sections, let Γ be the Cayley graph of Thompson's group F with respect to the generating set $A = \{x^{\pm 1}, y^{\pm 1}\}$ and let T be the tree associated to the set $\mathcal{N}$ of normal forms from Guba and Sapir's rewriting system Σ (described in Section 2.2).

We define a flow function $\Phi \colon \vec{E} \to \vec{P}$ as follows. Let e be a directed edge of Γ. If e lies on T, then $\Phi(e) := e$. Otherwise, by Lemma 2.3, we have $e = e_{\gamma, y^b}$ with

$$\gamma = p y^\epsilon x^i \in \mathcal{N},$$

where $b, \epsilon \in \{1, -1\}$, and $\gamma y^b \notin \mathcal{N}$. In this case, we define $\Phi(e)$ to be the directed path starting at e_- labeled by

$$\mathsf{label}(\Phi(e)) := \begin{cases} x^{-1} y^{-1} x y x^{-2} y x^2 & \text{if } b = 1 \text{ and } i > 2 \\ x^{-i} y^{-\epsilon} x^i y x^{-i-1} y^\epsilon x^{i+1} & \text{if } b = 1 \text{ and } 1 \leq i \leq 2 \\ x^{-2} y^{-1} x^2 y^{-1} x^{-1} y x & \text{if } b = -1 \text{ and } i > 3 \\ x^{-i} y^{-\epsilon} x^i y^{-1} x^{-i+1} y^\epsilon x^{i-1} & \text{if } b = -1 \text{ and } 2 \leq i \leq 3. \end{cases}$$

Note that the image of every directed edge e of Γ is a directed path from e_- to e_+. Hence the property of fixing the tree in the definition of flow function holds for the map Φ.

Next we show that the termination property holds for Φ, in order to complete the proof that Φ is a flow function. To do this, we use the well-founded strict partial order $\prec$ defined in Section 4. In particular, it suffices to prove the claim:

(∗) *For every pair of directed edges e, e' in Γ such that e is not in T and e' is in the path $\Phi(e)$, either e' lies in the tree T, or $e' \prec e$.*

To begin the proof of this claim, we note that for each edge e we have $\Phi(e^{-1}) = \Phi(e)^{-1}$, and every edge has the same weight as its inverse. Hence it suffices to prove this claim for any edge e (not in T) of the form $e = e_{\gamma,y}$.

Let g be the element of F with $g =_F e_-$, and write the normal form $\gamma := \mathsf{nf}(g)$ (using Lemma 2.3) in the form

$$\gamma = x^{i_n} y^{\epsilon_n} \cdots x^{i_1} y^{\epsilon_1} x^{i_0} = p y^{\epsilon_1} x^{i_0},$$

where each $\epsilon_j \in \{1, -1\}$, $n \geq 1$, and $i_0 \geq 1$. Let $g' =_F e_+$, and (again using Lemma 2.3) write the normal form $\gamma' := \mathsf{nf}(g')$ in the form

$$\gamma' = x^{j_{n'}} y^{\varepsilon_{n'}} \cdots x^{j_1} y^{\varepsilon_1} x^{j_0} = p' y^{\varepsilon_1} x^{j_0},$$

where each $\varepsilon_j \in \{1, -1\}$, $n' \geq 1$, and $j_0 \geq 2$.

Note also that Lemma 2.3 says that all edges in the path $\Phi(e)$ labeled x or x^{-1} are in the tree T. We consider the three other edges in $\Phi(e)$ labeled y or y^{-1}, in two cases.

Case I. *Suppose that $1 \leq i_0 \leq 2$.*

The first $y^{\pm 1}$ edge on $\Phi(e)$, namely $e_{py^{\epsilon_1}, y^{-\epsilon_1}}$, is the inverse of an edge on the normal form path in Γ starting at 1 and labeled by γ, and hence is in the normal form tree T.

A similar situation occurs for the third $y^{\pm 1}$ edge on $\Phi(e)$: Since $m(e) \geq 1$, Lemma 3.1 shows that the normal form $\gamma' = p' y^{\varepsilon_1} x^{j_0}$ labeling the path in Γ from 1 to e_+ satisfies $j_0 = i_0 + 1$ and $\varepsilon_1 = \epsilon_1$. Thus the third $y^{\pm 1}$ edge on $\Phi(e)$ is $e_{p', y^{\varepsilon_1}}$, which lies on the path labeled γ' in T.

Finally we consider the second $y^{\pm 1}$ edge e' on $\Phi(e)$. This edge has label y and its initial vertex $e'_- =_F p x^{i_0}$ has normal form $x^{i_n} y^{\epsilon_n} \cdots y^{\epsilon_2} x^{i_1 + i_0}$. If this edge is not in the tree, then Lemma 2.3 says that $n \geq 2$ and $i_1 + i_0 \geq 1$. Now the size sequence of e' is $\sigma(e') = (s_1, \ldots, s_{m-1})$, and so $W(e') < W(e)$; hence $e' \prec e$.

Case II. *Suppose that $i_0 \geq 3$.*

The function Φ replaces the edge e by the path from g to g' of length 9 labeled by $x^{-1} y^{-1} x y x^{-2} y x^2$. For easier discussion, consider the diagram in Figure 3. In that diagram, all dotted, double-tip edges, that is, the edges e, e_1, e_2, and e_3, are labeled by y and all other edges by x. The path $\Phi(e)$ connects g to g' by using the shortest path in the diagram, consisting of 9 edges, that does not use the edge e (thus, it uses the edges e_1^{-1}, e_2, and e_3, along with 6 $x^{\pm 1}$-edges which (as already noted above) are in the normal form tree T).

Lemma 6.1. *None of the edges e_1^{-1}, e_2, and e_3 is in the normal form tree, and each has weight strictly smaller than the weight of e.*

FIGURE 3. e and $\Phi(e)$ when $i_0 \geq 3$

Proof. Let $s(\gamma) = (s_n, \ldots, s_0)$ be the sequence of partial sums of the x-exponents in γ, and let $m := m(e) = \min\{k \mid s_k \leq 0\}$. Denote $f := gx^{-1}y^{-1}$ and $h := fx =_F gx^{-1}y^{-1}x$. Note that

$$\mathsf{nf}((gx^{-1})) = x^{i_n}y^{\epsilon_n} \cdots x^{i_1}y^{\epsilon_1}x^{i_0-1}.$$

Since $i_0 - 1 \geq 2$, the edge e_1^{-1} is not on the normal form tree. But this means that the normal form of f ends with at least one x (in fact, exactly $i_0 - 2$ according to Lemma 3.3) and the normal form of h ends with at least two (in fact $i_0 - 1$), which implies that e_2 is not on the normal form tree. The normal form of g' ends with at least 4 letters x (Lemma 3.1 says g' ends with exactly $i_0 + 1$), which implies that $g'x^{-2}$ ends with at least 2 (exactly $i_0 - 1$), and so e_3 is not on the normal form tree either.

We have

$$m_1 := m(e_1^{-1}) = m'(\mathsf{nf}(gx^{-1})) = \min\{k \mid i_k + \cdots + (i_0 - 1) \leq 1\} = \min\{k \mid s_k \leq 2\}$$

when the set is nonempty and, otherwise, $m_1 = n$. Hence $m_1 \leq m$. Now the vector $s(\mathsf{nf}(gx^{-1}))$ of cumulative x-exponents is $s(\mathsf{nf}(gx^{-1})) = (s_n - 1, \ldots, s_0 - 1)$, and so Lemmas 3.3 and 4.1 say that the sequence of rule sizes associated to the edge e_1 is $\sigma(e_1) = \sigma(e_1^{-1}) = (s_0 - 2, \ldots, s_{m_1-1} - 2)$. Now $s_0 = i_0 \geq 3$ (since we are in Case II), and so $m_1 \geq 1$; moreover, for each $k = 0, \ldots, m_1 - 1$ we have $s_k \geq 3$. Using the fact that C is a nondecreasing sequence of positive integers (for the inequalities) and the recurrence relation defining C (in the last equality) yields

$$W(e) - W(e_1) = \sum_{k=0}^{m-1} C(s_k) - \sum_{\ell=0}^{m_1-1} C(s_\ell - 2) = \sum_{k=0}^{m_1-1} [C(s_k) - C(s_k - 2)] + \sum_{k=m_1}^{m-1} C(s_k)$$

$$\geq C(s_0) - C(s_0 - 2) = C(s_0 - 1) + C(s_0 - 2) + 2 > 0.$$

Therefore $W(e) > W(e_1)$.

By using Table 2 we can calculate the cumulative x-exponents vector $s(f)$ directly from the cumulative x-exponents vector $s(gx^{-1})$, and from there we compute the cumulative x-exponents vector $s(h)$ simply by adding 1 in each coordinate of $s(f)$. Partial results are given in Table 3 (with the no y^{-1}-cancellation case on the left and the y^{-1}-cancellation case on the right); the cancellation cases in that table refer to crossing the edge e_1^{-1}, that is multiplying gx^{-1} by y^{-1}.

k	$s(f)_k$	$s(h)_k$	$s(g)_k$
0	$s_0 - 2$	$s_0 - 1$	s_0
1	$s_1 - 2$	$s_1 - 1$	s_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m_1 - 1$	$s_{m_1-1} - 2$	$s_{m_1-1} - 1$	s_{m_1-1}
m_1	$\mathbf{0}$	1	s_{m_1}
$m_1 + 1$	$s_{m_1} - 1$	s_{m_1}	s_{m_1+1}
$m_1 + 2$	$s_{m_1+1} - 1$	s_{m_1+1}	s_{m_1+2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	$s_{m-1} - 1$	s_{m-1}	$\mathbf{s_m}$
$m + 1$	$s_m - 1$	$\mathbf{s_m}$	

k	$s(f)_k$	$s(h)_k$	$s(g)_k$
0	$s_0 - 2$	$s_0 - 1$	s_0
1	$s_1 - 2$	$s_1 - 1$	s_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m_1 - 1$	$s_{m_1-1} - 2$	$s_{m_1-1} - 1$	s_{m_1-1}
m_1	$s_{m_1+1} - 1$	s_{m_1+1}	s_{m_1}
$m_1 + 1$	$s_{m_1+2} - 1$	s_{m_1+2}	s_{m_1+1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m - 2$	$s_{m-1} - 1$	s_{m-1}	s_{m-2}
$m - 1$	$s_m - 1$	$\mathbf{s_m}$	s_{m-1}
m			$\mathbf{s_m}$

TABLE 3. Cumulative x-exponents of f, h, and g: the no y^{-1}-cancellation case (left) and the y^{-1}-cancellation case (right)

We claim that, in each $s(h)_k$ and $s(g)_k$ column in Table 3, the boldface entry is the first nonpositive entry. For g this is immediate from the definition of the cutoff point $m(g) = m$. For $k = 0, \ldots, m_1 - 1$, by the definition of m_1, we have $s_k \geq 3$, which implies that $s(h)_k = s_k - 1 \geq 2$. By the definition of m, we also have that $s_k \geq 1$ for $k \leq m - 1$ and this completes our claim. Note that we just proved that $m(h) = m + 1$ in the no y^{-1} cancellation case, and $m(h) = m - 1$ in the y^{-1}-cancellation case.

Since the cumulative x-exponents of g and h before the index $m = m(g)$ and $m(h)$, respectively, represent the rule sizes associated to the edges e and e_2, we again use them to compare weights. In particular, the difference in weights $W(e) - W(e_2)$ is the difference between the sum of the entries in the sequence C (defined in Section 4.2) with indices in the $s(g)_k$ column before $m(g)$ and the sum of the C sequence entries with indices in the $s(h)_k$ column before $m(h)$; hence entries appearing in both columns will cancel. In more detail, in the no y^{-1} cancellation case we have

$$
\begin{aligned}
W(e) - W(e_2) &= \sum_{k=0}^{m-1} C(s_k) - \sum_{\ell=0}^{m(h)-1} C(s(h)_\ell) \\
&= \sum_{k=0}^{m_1-1} [C(s_k) - C(s_k - 1)] + C(s_{m_1}) - C(1) + \sum_{k=m_1+1}^{m-1} [C(s_k) - C(s_{k-1})] - C(s_{m-1}) \\
&= \sum_{k=0}^{m_1-1} [C(s_k) - C(s_k - 1)] - 1 \\
&\geq C(s_0) - C(s_0 - 1) - 1 = 2C(s_0 - 2) + 2 - 1 > 0
\end{aligned}
$$

where in the inequalities we used that C is a nondecreasing sequence of positive integers, and between the two inequalities we apply the recurrence relation defining

the sequence C. A similar telescoping sum occurs in the y^{-1} cancellation case, yielding

$$W(e) - W(e_2) = \sum_{k=0}^{m_1-1} [C(s_k) - C(s_k - 1)] + C(s_{m_1}) \geq C(s_0) - C(s_0 - 1) > 0.$$

Therefore $W(e) > W(e_2)$.

Finally, since $i_0 \geq 3$, Lemma 3.1 says that the normal form $\mathsf{nf}(g')$ has a suffix x^{j_0} with $j_0 \geq 4$, and so the normal form $\mathsf{nf}(g'x^{-2})$ is obtained by removing an x^2 suffix from $\mathsf{nf}(g')$. Hence $s(g'x^{-2})_k = s(g')_k - 2$ for all k, and so $m(e_3) \leq m(e)$. Then the size sequence $\sigma(e)$ contains at least as many entries as $\sigma(e_3)$, and has larger corresponding entries. Therefore $W(e) > W(e_3)$.

This completes the proof of Case II and Lemma 6.1. $\qquad\square$

This also completes the proof of the claim $(*)$ (on page 17). Therefore the function Φ satisfies the termination property and is a flow function for Thompson's group F. This flow function is bounded, with bounding constant $K = 13$.

It remains to show that the set $\mathsf{graph}\,(\Phi)$ is a regular language. For the flow function Φ, this graph is

$$
\begin{aligned}
\mathsf{graph}\,(\Phi) \;=\; & \cup_{a\in A}\left\{(u,a,a) \mid ua \in \mathcal{N} \text{ or } u \in (\mathcal{N}\cap A^*a^{-1})\right\} \\
& \cup \left(\cup_{\epsilon\in\{1,-1\}} \cup_{i\in\{1,2\}} \left\{(uy^\epsilon x^i, y, x^{-i}y^{-\epsilon}x^i yx^{-i-1}y^\epsilon x^{i+1}) \mid uy^\epsilon x^i \in \mathcal{N}\right\}\right) \\
& \cup \left(\cup_{\epsilon\in\{1,-1\}} \cup_{i\in\{2,3\}} \left\{(uy^\epsilon x^i, y^{-1}, x^{-i}y^{-\epsilon}x^i y^{-1}x^{-i+1}y^\epsilon x^{i-1}) \mid uy^\epsilon x^i \in \mathcal{N}\right\}\right) \\
& \cup \left\{(ux, y, x^{-1}y^{-1}xyx^{-2}yx^2) \mid u \in \mathcal{N}\cap(A^*y^{\pm1}A^*x^2)\right\} \\
& \cup \left\{(ux^2, y^{-1}, x^{-2}y^{-1}x^2y^{-1}x^{-1}yx) \mid u \in \mathcal{N}\cap(A^*y^{\pm1}A^*x^2)\right\} \\
=\; & \cup_{a\in A}\left(\mathcal{N}_a \cup (\mathcal{N}\cap A^*a^{-1})\right) \times \{a\} \times \{a\} \\
& \cup \left(\cup_{\epsilon\in\{1,-1\}} \cup_{i\in\{1,2\}} (\mathcal{N}\cap A^*y^\epsilon x^i) \times \{y\} \times \{x^{-i}y^{-\epsilon}x^i yx^{-i-1}y^\epsilon x^{i+1}\}\right) \\
& \cup \left(\cup_{\epsilon\in\{1,-1\}} \cup_{i\in\{2,3\}} (\mathcal{N}\cap A^*y^\epsilon x^i) \times \{y^{-1}\} \times \{x^{-i}y^{-\epsilon}x^i y^{-1}x^{-i+1}y^\epsilon x^{i-1}\}\right) \\
& \cup \left(\mathcal{N}\cap(A^*y^{\pm1}A^*x^3)\right) \times \{y\} \times \{x^{-1}y^{-1}xyx^{-2}yx^2\} \\
& \cup \left(\mathcal{N}\cap(A^*y^{\pm1}A^*x^4)\right) \times \{y^{-1}\} \times \{x^{-2}y^{-1}x^2y^{-1}x^{-1}yx\}.
\end{aligned}
$$

Since the language $\mathcal{N}$ is regular, Lemma 2.1 says that the quotient language $\mathcal{N}_a$ is also regular. Now closure of the class of regular languages under union, intersection, concatenation, etc., shows that the left factor in each triple is a regular language. Since any set consisting of a single word is also regular, Lemma 2.1 shows that each of the above products of three regular languages is also regular. Finally, closure under union again shows that $\mathsf{graph}\,(\Phi)$ is regular.

Therefore Φ is a regular bounded flow function, and so Thompson's group F is autostackable. The prefix-rewriting system in the statement of Theorem 1.1 is the system associated to the flow function Φ, and hence is convergent. This completes the proof of Theorem 1.1.

Acknowledgments

References

[1] Bleak, C., Brough, T., and Hermiller, S., *Determining solubility for finitely generated groups of PL homeomorphisms*, arXiv:1507.06908.

[2] M. Bridson, *The geometry of the word problem*, In: Invitations to geometry and topology, Oxf. Grad. Texts Math. **7**, pp. 29–91, Oxford Univ. Press, Oxford, 2002.

[3] Brittenham, M., Hermiller, S. and Holt, D.F., *Algorithms and topology of Cayley graphs for groups*, J. Algebra **415** (2014), 112-136.

[4] Brown, K.S. and Geoghegan, R., *An infinite-dimensional torsion-free FP_∞ group*, Invent. Math. **77** (1984), no. 2, 367-381.

[5] Burillo, J., Matucci, F. and Ventura, E., *The conjugacy problem in extensions of Thompson's group F*, Israel J. Math. **216** (2016), no. 1, 15-59.

[6] Cleary, S., Hermiller, S., Stein, M. and Taback, J., *Tame combing and almost convexity conditions*, Math. Z. **269** (2011), 879-915.

[7] Epstein, D.B.A., Cannon, J., Holt, D., Levy, S., Paterson, M. and Thurston, W., Word Processing in Groups, Jones and Bartlett, Boston, 1992.

[8] Elder, M. and Taback, J., *Thompson's group F is 1-counter graph automatic*, Groups Complex. Cryptol. **8** (2016), no. 1, 21–33.

[9] Golan, G., *The generation problem in Thompson group F*, arxiv:1608.02572.

[10] Golan, G. and Sapir, M., *On subgroups of R. Thompson's group F*, Trans. Amer. Math. Soc. **369** (2017), 8857-8878.

[11] Guba, V.S. and Sapir, M., *The Dehn function and a regular set of normal forms for R. Thompson's group F*, J. Austral. Math. Soc. Ser. A **62** (1997), no. 3, 315-328.

[12] Guba, V.S., *The Dehn function of Richard Thompson's group F is quadratic*, Invent. Math. **163** (2006), 313-342.

[13] Guba, V. and Sapir, M., Diagram groups, Mem. Amer. Math. Soc. **130** (1997), no. 620.

[14] Hopcroft, J. and Ullman, J.D., Introduction to automata theory, languages, and computation, Addison-Wesley Series in Computer Science, Addison-Wesley Publishing Co., Reading, Mass., 1979.

[15] Kassabov, M. and Matucci, F., *The simultaneous conjugacy problem in groups of piecewise linear functions*, Groups Geom. Dyn. **6** (2012), no. 2, 279-315. '

[16] Lyndon, R.C. and Schupp, P.E., Combinatorial group theory, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[17] Otto, F., *On s-regular prefix-rewriting systems and automatic structures*, Computing and combinatorics (Tokyo, 1999), 422-431, Lecture Notes in Comput. Sci. **1627**, Springer, Berlin, 1999.

[18] *Thompson's group at 40 years: Preliminary problem list*, AIM Research Conference Center, January 2004, available from
http://www.aimath.org/WWN/thompsonsgroup/thompsonsgroup.pdf.

Department of Mathematics and Statistics, Villanova University, 800 Lancaster Avenue, Villanova, PA 19085, USA
    E-mail address: nathan.corwin@villanova.edu

Department of Mathematics, Ben-Gurion University of the Negev, P.O.B. 653, Be′er Sheva 84105, Israel
    E-mail address: golangi@bgu.ac.il

Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0130, USA
    E-mail address: hermiller@unl.edu

Department of Mathematics, University of North Alabama, Florence, AL 35632, USA
    E-mail address: ajohnson18@una.edu

Department of Mathematics, Hofstra University, Hempstead, NY 11549, USA
    E-mail address: zoran.sunik@hofstra.edu