Rewriting systems and Noncommutative Gröbner bases
Twenty-second Holiday Mathematics Symposium
A B S T R A C T S

****

Geir Agnarsson, University of California at Berkeley
On Functionals and Outside Corners of Monomial Ideals

Let k be a field, R a free k-algebra on some set, and I a two-sided ideal of R. We will present sufficient conditions on the ideal I and the algebra R for I to have a finite basis of functionals $R \to k$. Thereby we get a generalization of F.S. Macaulay's result from 1916, that every ideal I of $k[X_1, \ldots, X_n]$ has a finite basis of functionals $k[X_1, \ldots, X_n] \to k$. In this case we get moreover that if I has a Gröbner basis consisting of p elements, then the number of functionals in such a basis is, for fixed n, bounded by a $\Theta(p^{[n/2]})$-function. This function is closely related to the maximal number of corners of a monomial ideal generated by p monomials in n variables.

****

Mike Bardzell, Salisbury State University
A Lifted Gröbner Basis for the Enveloping Algebra

In this talk we consider a finite-dimensional quotient of a path algebra and demonstrate how its enveloping algebra can be modeled on a computer for homological calculations. A method of lifting a reduced Gröbner basis from the original algebra to its enveloping algebra is also presented.

****

Lidia Filus, Northeastern Illinois University
Why Gröbner Bases? Comparison of some methods for solving systems of nonlinear equations

Comparisons with some of the fixed point algorithms will be discussed. Questions regarding possibilities to apply Gröbner Bases ideas to these algorithms will be raised.

****

Tatiana Gateva-Ivanova, American University in Bulgaria
Noncommutative Gröbner Bases in Skew-polynomial Rings

We study a class of graded standard finitely presented quadratic algebras A over a fixed field K, called binomial skew polynomial rings. Following a classical tradition (and recent trend), we take a combinatorial approach to study the algebraic properties of A. Our purpose is to exhibit a class of non-commutative algebras with good algebraic properties among which Noetherianness, being a domain, regularity, which can be read off a presentation $A = K\langle X\rangle/(F)$, where $X = \{x_1, ..., x_n\}$ is a set of indeterminates of degree 1, $K\langle X\rangle$ is the unitary free associative algebra generated by X and (F) is the two-sided ideal, generated by a finite set of homogeneous polynomials.

Following [2], a **binomial skew polynomial ring** is a graded algebra $A = K\langle X\rangle/(F)$, which has precisely $n(n-1)/2$ defining relations $F = \{x_j x_i - c_{ij} u_{ij} \mid 1 \leq i < j \leq n\}$, such that (a) c_{ij} are non-zero coefficients; (b) $u_{ij} = x_{i'} x_{j'}$, for $1 \leq i < j \leq n$, with $i' < j$, $i' < j'$; (c) the set F is a reduced Gröbner basis (w.r.t. the deg-lex ordering on the free semigroup $\langle X\rangle$) of the ideal (F).

We give necessary and sufficient condition for Noetherianness of A in case that the set of relations F is square-free. We show that the notion of Gröbner basis can be extended for modules over strictly ordered algebras, so that an analogue of Bergman's Diamond Lemma holds (cf. [2], 2.2) and apply this for Noetherian skew polynomial rings with binomial relations. The following result can be extracted from [2] and [3].

Theorem *Let $A = K\langle X\rangle/(F)$, be a binomial skew-polynomial ring. Suppose the set of relations F is square-free. Then the following conditions are equivalent:*

(1) each $x_i x_j$, with $1 \leq i < j \leq n$, appears in the right hand side of some relation in F;

(2) A is left Noetherian;

(3) A is right Noetherian;

(4) Any left ideal of A has a finite Gröbner basis (with respect to some grading of A;)

(5) Any right ideal of A has a finite Gröbner basis (with respect to some grading of A; Furthermore, any of the conditions (1) - (5) implies that

(a) the membership problem for finitely generated one-sided ideals in A is decidable;

(b) A is a domain;

(c) A is regular in the sense of Artin-Schelter (cf.[2]).

(d) the operator on X^2 naturally defined by the quadratic relations F satisfies the setheoretic Yang Baxter equations.

References

[1] M. Artin, W. Schelter, Graded algebras of global dimension 3, *Advances of Math.* **66** (1987), 171-216.

[2] T. Gateva-Ivanova, Skew polynomial rings, *J. Algebra,* **185** (1996), 710-753.

[3] T. Gateva-Ivanova, M. Van den Bergh, Semigroups of I-type *Preprint.*

****

Susan Hermiller, New Mexico State University

Rewriting systems and 3-manifolds

The fundamental groups of most (conjecturally, all) closed 3-manifolds with uniform geometries have finite complete rewriting systems. The fundamental groups of a large class of amalgams of circle bundles also have finite complete rewriting systems. The general case remains open. This is joint work with Mike Shapiro.

****

Anne Heyworth, University of Wales

Computerised left Knuth-Bendix procedure for enumerating cosets

Using the presentation ⟨X,R⟩ of a group G, and the generators of a subgroup H in G, and a monomial ordering on F(X) one can do a left Knuth-Bendix completion. This gives a "left-complete" system that can be applied to the words in the free group F(X) to enumerate the right cosets of H in G. I have computerised the process (using Axiom). It is a valid alternative to the Todd-Coxeter process.

****

Xenia Kramer, New Mexico State University
The Noetherian Property in Some Quadratic Algebras

We introduce a new class of noncommutative rings called *pseudopolynomial rings* and give sufficient conditions for such a ring to be Noetherian. Pseudopolynomial rings are standard finitely presented algebras over a field with some additional restrictions on their defining relations—namely that the polynomials in a Gröbner basis for the ideal of relations must be homogeneous of degree 2—and on the Ufnarovskii graph $\Gamma(A)$. The class of pseudopolynomial rings properly includes the generalized skew polynomial rings introduced by M. Artin and W. Schelter. We use the graph $\Gamma(A)$ to define a weaker notion of almost commutative, which we call *almost commutative on cycles.* We show as our main result that a pseudopolynomial ring which is almost commutative on cycles is Noetherian. A counterexample shows that a Noetherian pseudopolynomial ring need not be almost commutative on cycles.

Xenia Kramer, New Mexico State University
Combinatorial Homotopy of Simplicial Complexes

Combinatorial homotopy is an analog of the usual homotopy theory of algebraic topology which is intended to reflect combinatorial features of a simplicial complex. The combinatorial version of a path is a finite sequence of simplices $\sigma_1, \sigma_2, \ldots, \sigma_n$ in a complex Δ where for $1 \leq i < n$, the dimension of $\sigma_i \cap \sigma_{i+1}$ is at least q, for some previously set value q. Two paths are homotopic in this combinatorial sense if one can be deformed to the other while keeping the dimension of the intersection of adjacent simplices at least q.

The motivation for the study of combinatorial homotopy comes from the analysis of complex information systems. The flow of traffic in such a system is constrained by the underlying structure of the system; that is, the dynamic behavior of a system is influenced by its static structure. Combinatorial homotopy was introduced to describe features of that static structure.

In this talk, we will focus on the mathematics of combinatorial homotopy. After defining it, we give an algorithm for computing it which uses Gröbner bases in a free associate algebra. This work was done in collaboration with Reinhard Laubenbacher.

****

Peter Malcolmson, Wayne State University
Gröbner-Shirshov bases for quantum enveloping algebras

We give a method for finding Gröbner-Shirshov bases for the quantum enveloping algebras of Drinfel'd and Jimbo, show how the methods can be applied to Kac-Moody algebras, and explicitly find the bases for quantum enveloping algebras of type A (for the quantizing coefficient not an eighth root of unity).

****

Eduardo do Nascimento Marcos, IME-USP (Dept. de Matematica) Brazil
Graded Rings of Local Finite Representation Type

Graded Artin algebras whose category of graded modules is locally of finite representation type are introduced. The representation theory of such algebras is studied. In the hereditary case and in the stably equivalent to hereditary case, such algebras are classified.

****

Mike Newman, Australian National University
Some use of Knuth-Bendix in the study of groups

I will describe the use of Knuth-Bendix and other software for proving that certain Engel groups are nilpotent.

****

Gretchen Ostheimer, Tufts University
Finding Matrix Representations for Polycyclic Groups

It has been known since the 1960's that a solvable group G can be embedded in $GL(n, Z)$ for some n if and only if G is polycyclic. In 1990, Segal described an algorithm for constructing such an embedding when the polycyclic group is given by a finite presentation; however, that algorithm is not suitable for computer implementation. We describe an alternative algorithm for this problem. Our primary tool is a program developed by Lo in 1996 in which the Gröbner basis method from commutative ring theory is extended to the group ring of a polycyclic group given by a finite presentation. Preliminary experiments indicate that our algorithm will be efficient enough to investigate some interesting examples. Further experimentation is needed to determine the range of input for which the algorithm is practical with current technology.

This work was undertaken as part of the author's Ph.D. thesis under the direction of Charles Sims and was continued collaboratively with Eddie Lo.

****

Deepak Parashar, The Robert Gordon University
Quantum Groups and Quantum Spaces

During the recent past there has been considerable success in the development of the theory of Quantum Groups, an exciting new generalization of ordinary Lie Groups. These mathematical structures have a wide variety of applications, primarily in non-commutative geometry and models in two-dimensional physics. The purpose of this talk is to give a systematic mathematical introduction to the subject of quantum groups and quantum spaces using GLq(2) as our main example.

****

F. Leon Pritchard, Rutgers University-Newark
The Ideal Membership Problem in Non-Commutative Polynomial Rings

Let X be a non-commutative monoid with term order; let R be a commutative, unital ring; let I be an ideal in the non-commutative polynomial ring $R\langle X\rangle$; and let $f \in R\langle X\rangle$. In this setting the problem of determining whether $f \in I$ is studied. In a manner analogous to the commutative case, weak Gröbner bases are defined and their basic properties are studied. We will see that in the non-commutative setting, when the coefficient ring is not a field, and when we enlarge the polynomial ring by adding more variables, weak Gröbner bases may exhibit unpleasant behavior that has no analog in the commutative case. Quite in general for $f \in R\langle X\rangle$, it is undecidable whether $f \in I$. This follows from the fact that the word problem for free semigroups is undecidable. If I is generated by a recursively enumerable set, then we give a semi-decision procedure that halts if and only if $f \in I$. Finally we examine a class of nicely behaved ideals for which weak Gröbner bases can be easly computed.

****

Mark Stankus, California Polytechnic State University Of San Luis Obispo
Applying Noncommutative Gröbner Basis to Problems in Analysis and Control Theory

There are a number of theorems in analysis and control theory which have a major algebraic component. This algebraic component consists of finding a system of matrix equations which has some desirable property (e.g., amenable to numerical solution) and which is equivalent to a given system of matrix equations.

Our experience has shown that transforming the given system of matrix equations into one which has a desirable property cannot be done by using a noncommutative Gröbner Basis algorithm alone. One common concern is that a noncommutative Gröbner Basis can be infinite and so the noncommutative Gröbner Basis algorithm will never terminate. When one stops the algorithm, a Gröbner Basis is not obtained. Whether or not the set of relations obtained is a Gröbner Basis, many of the relations obtained may be conceptually redundant from the perspective of the application being considered. For this reason, we consider the following additional algorithm:

(1) Find a minimal generating set for an ideal (so that one can remove conceptually redundant information generated by the Gröbner Basis algorithm).

Since allowing for changes of variables can be very useful, we also consider the following additional algorithm which facilitates the introduction of new variables:

(2) Compute decompositions of a matrix expression into a composition of matrix expressions which respect certain qualitative information.

Our methodology for deriving the algebraic component of a theorem uses the noncommutative Gröbner Basis algorithm and the two algorithms mentioned above. The methodology consists of running these three algorithms on a computer for a finite amount of time, displaying the relations obtained, having a person view the relations obtained, making any desired conclusions based on the non-algebraic components of the theorem, making any desired changes of variables and repeating the process as necessary.

****

Victor Ufnarovski, Lund University
BERGMAN and ANICK - programs for non-commutative algebras

The last versions of two Computer Algebra programs are discussed. One of them (author - J.Backelin, Stokholm University) is a rather efficient program for calculating Gröbner bases for graded algebras. Another one (authors -

A.Podoplelov, V.Ufnarovski, Institute of Mathematics, Moldova; LTH, Lund, Sweden) calculates Anick's resolution.

****

John J. Wavrik, University of California-San Diego

Gröbner Bases, Rewrite Rules, and Matrix Expressions

This talk concerns the automated simplification of expressions which involve non-commuting variables. The technology has been applied to the simplification of matrix and operator theory expressions which arise in engineering applications. The non-commutative variant of the Gröbner Basis Algorithm is used to generate rewrite rules. We will also look at the phenomenon of infinite bases and implications for automated theorem proving.

****

Sergey Yuzvinsky, University of Oregon

Koszul algebras related to hyperplane arrangements and graphs

In the talk, we discuss the Orlik-Solomon algebras. Such an algebra A is a combinatorial interpretation of the cohomology algebra of the complement of a complex hyperplane arrangement. A has a standard, so called "broken circuit" basis. For the well-studied class of supersolvable arrangements this basis has a very nice property that allows us to deform A to a monomial algebra A_0 defined by a graph. Studying a basis of A_0 we recover that it is Koszul (a theorem by Fröberg) and obtain an elegant formula for its Hilbert series. Then Drinfeld's deformation theory implies that A is Koszul.

****